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Right Engel Subgroups

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Summary

In this thesis we find deep results on the structure of normal right n-Engel subgroups that are contained in some term of the upper central series of a group. We start with some known results, and one new result, on the structure of locally nilpotent n-Engel groups. These are closely related to the solution of the restricted Burnside problem. We also give specific details of the structure of 2-Engel and 3-Engel groups in the context of these results. The main idea of this thesis is to generalise these results to apply to normal upper central right n-Engel subgroups. We also consider the special case of locally finite p-groups and again generalise some deep results on the structure of n-Engel such groups to apply to right n-Engel subgroups. For each of the theorems on right n-Engel subgroups, complete details are given for the case $n = 2$. Right 3-Engel subgroups have a more complicated structure. For these we prove a Fitting result, for which we exclude the prime 3, and using this we also find a sharp bound on the upper central degree in the torsion-free case. In fact we only need to exclude the primes 2, 3 and 5 for this result. This gives some further information on the structure of right 3-Engel subgroups in the context of the main theorems.

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Contents

Chapter 1

Introduction

1.1 Engel groups and the Burnside problems

The commutator of two elements x and y in a group is defined as $[x, y] = x^{-1}y^{-1}xy$. We use the left normed convention for commutators of more than two elements. Thus $[x_1, \ldots, x_{m+1}] = [[x_1, \ldots, x_m], x_{m+1}]$. Let n be a non-negative integer. The n-Engel word $[x,_n y]$ is defined recursively by $[x,_0 y] = x$ and $[x,_{n+1} y] = [[x,_n y], y]$. If every pair of elements in a group satisfies an n-Engel word, for some n dependent on the pair, then the group is said to be an Engel group. If every pair satisfies an n-Engel word with n independent of the pair, then the group is an n-Engel group. It is clear that a nilpotent group of nilpotency class c is a c-Engel group. Since only a pair of elements are involved in the n-Engel word, it is also true that locally nilpotent groups are Engel groups. However, the converse is not true, as an example of Golod [13] shows. Thus Engel groups are a class of generalised locally nilpotent groups.

Engel groups have their origin in a paper of Burnside [7] from 1901, from which the famous Burnside problems also originated. In particular, Burnside was interested in whether finitely generated groups of bounded exponent are necessarily finite. He proved that this was the case for groups of exponent 3 and noted that every element in these groups commuted with its conjugates. It follows that groups of exponent 3 are 2-Engel groups. The 1-Engel groups are exactly the abelian groups, and it is clear that finitely generated abelian groups of bounded exponent are finite. Thus it would be natural to assume that it is the fact that groups of exponent 3 are 2-Engel groups that makes them locally finite. Indeed Burnside wrote a follow on paper [8] in which he studies 2-Engel groups and shows that 2-Engel groups of bounded exponent are locally finite.

In the context of periodic groups (i.e. groups in which every element has finite order), every locally nilpotent group is locally finite and in fact Burnside proved that periodic 2-Engel groups are locally nilpotent. Later Hopkins [23] proved that 2-Engel groups are nilpotent of class at most 3. For each of the Burnside problems we have analogous problems for Engel groups. The main problems arising from the work of Burnside are:

The general Burnside problem. Is every periodic group locally finite?

The Burnside problem. Is every group of bounded exponent locally finite?

The restricted Burnside problem. For fixed positive integers r and n, is there a largest finite r-generator group with exponent n?

The analogous statements for Engel groups are:

The general local nilpotence problem. Is every Engel group locally nilpotent?

The local nilpotence problem. For each positive integer n, is every n-Engel group locally nilpotent?

The restricted local nilpotence problem. For fixed positive integers r and n, is there a largest nilpotent r-generator n-Engel group?

Golod's examples [13] give counterexamples for both the general Burnside and general local nilpotence problems. The restricted Burnside problem was reduced to the case of prime-power exponent by P. Hall and Higman [20], and this was then solved by Zel'manov [41, 42]. The restricted local nilpotence problem is also known to have a positive answer and follows from the work of Zel'manov and a theorem due to Wilson [38]. There are well-known counterexamples to the Burnside problem, which was first shown to be false by Novikov and Adjan [31, 32, 33]. However, the local nilpotence problem is still open. It is however known to be positive for $n \leq 4$. The case $n = 2$ was shown by Hopkins [23], the case $n = 3$ by Heineken [22] and the case $n = 4$ by Havas and Vaughan-Lee [21]. There are also some situations where n-Engel groups for general n are known to be locally nilpotent. The first general result on Engel groups was a theorem of Zorn [43], which states that every finite Engel group is nilpotent. It was shown by Gruenberg [15] that solvable n-Engel groups are locally nilpotent.

1.2 Engel elements and subgroups

An element a in a group G is said to be right Engel if $[a,_{n(g)} g] = 1$ for all $g \in G$. If n can be chosen independently of g, then we say that a is a right n-Engel element. Thus G is an Engel group if every element of G is right Engel and G is an n-Engel group if every element of G is right n-Engel. Clearly for every element a in the $(n+1)$th term of the upper central series, $Z_n(G)$, we have that all the elements in $\langle a \rangle^G$ are right n-Engel. It is conversely not true in general that right n-Engel elements need to be in the hypercentre, $\cup_{i=1}^{\infty} Z_i(G)$. As with n-Engel groups we do have that the right n-Engel elements are in the hypercentre for finite groups [4] and finitely generated solvable groups [5].

The right n-Engel elements of a group need not form a subgroup. We will be interested in the structure of normal subgroups consisting of right n-Engel elements.

Definition 1.1. Let H be a subgroup of a group G. Then H is said to be a *right n-Engel subgroup* if all the elements of H are right n-Engel elements of G.

One can define left n-Engel elements and subgroups in the obvious way, however these are not that interesting in the context of the results in this thesis.

1.3 An overview of this thesis

In this thesis we will study the structure of n-Engel groups and right n-Engel subgroups. We will first discuss some background material in Chapter 2, which we will need throughout the thesis. We then begin Chapter 3 by stating some theorems on the structure of nilpotent n-Engel groups, one of which is new and a proof is given. These theorems give rise to some integer valued functions, which we then evaluate for $n \leq 3$. We then look at locally finite p-groups that are n-Engel, stating similar structure theorems which again give rise to some integer valued functions. These are then evaluated for $n = 2$ and some partial results are given for $n = 3$. We then generalise each of the theorems in Chapter 4. In particular, the first of these theorems gives a positive solution to a generalised version of the restricted local nilpotence problem for right n-Engel subgroups. The theorems in Chapter 4 also give rise to integer valued functions. However, these are more difficult to evaluate. We therefore have a seperate chapter for the case $n = 2$ in which all the values are found. Much of Chapters 3 - 5 appears in three joint papers with Gunnar Traustason [9, 10, 11]. We then in Chapter 6 consider right 3-Engel subgroups and prove some results on their structure. In Chapter 7 we discuss some further work that would follow on from the work in this thesis. We end with an appendix in which we place some calculations related to Chapter 6.

Chapter 2

Background material

In this chapter we discuss some background material, in particular focussing on material which will be used for proofs later in this thesis.

2.1 Free groups and varieties of groups

One type of group that we will make use of both in proofs and examples is a free group. This is useful as every group is a quotient of a free group, which can be seen from the definition.

Definition 2.1. A group F is a *free group with basis* $X \subseteq F$ if, for every group G and function $f : X \mapsto G$, there is a unique homomorphism $\phi : F \mapsto G$ such that ϕ extends f.

One can construct a free group with basis X as the set of reduced words on X. That is, the set of words $x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ and no $x \in X$ appears adjacent to x^{-1}. One can see, by considering f as the inclusion map from X to this group, that this is the only free group with basis X, up to isomorphism. Free groups are particularly useful when dealing with classes of groups known as varieties.

Definition 2.2. A *variety of groups* is a class of groups G satisfying a given set of relations of the form

$$f(x_1, \ldots, x_i) = 1$$

for every $x_1, \ldots, x_i \in G$.

Thus, for example, the class of n-Engel groups is a variety, where the relations are

$$f(x, y) = [x, \underbrace{y, \ldots, y}_{n}] = 1.$$

Note that in a variety there is a largest group with rank at most m, where m is the cardinality of some set, in the sense that every other group with rank at most m in the variety is isomorphic to a quotient of this group. To see this, take the free group with basis of size m and quotient out by the normal closure of all the elements which are trivial in the relations. We refer to this as the relatively free group on m generators in the variety. The following theorem will allow us to see that there is a relatively free locally nilpotent n-Engel group on any set of generators, which in particular gives the positive solution to the restricted local nilpotence problem.

Theorem 2.3. *Let $\mathcal{V}$ be a variety of groups and $\mathcal{LV}$ be the class of locally nilpotent groups in $\mathcal{V}$. Then the following are equivalent.*
(i) $\mathcal{LV}$ is a subvariety of $\mathcal{V}$.
(ii) Every finitely generated residually nilpotent group in $\mathcal{V}$ is nilpotent.
(iii) For each $d \in \mathbb{N}$ there is an upper bound on the nilpotency class of d-generator nilpotent groups in $\mathcal{V}$.

A theorem of Wilson [38], which we will generalise for right n-Engel subgroups in Chapter 4, shows that condition (iii) holds when $\mathcal{V}$ is the variety of n-Engel groups.

2.2 Properties of commutators

In this section we discuss some basic properties of commutators, which we will use throughout this thesis. For group elements x and y, we refer to $[x, y] = x^{-1}y^{-1}xy$ as the commutator of x and y, or x commuted by y. The set of commutators with entry set X, where X is a subset of a group G, is defined as the smallest set containing X and the identity element of G, such that the set is closed under taking commutators. A commutator with entry set X of the form $[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n \in X$, will be referred to as a left normed commutator. The number of entries in a commutator is referred to as the weight of the commutator. We say that a commutator has multi-weight $(i_1, i_2, \ldots)$ in $(x_1, x_2, \ldots)$, where these lists are the same size, if it has entry set some subset of $\{x_1, x_2, \ldots\}$ and has i_j entries of x_j for $j = 1, 2, \ldots$.

From the definition of a commutator we have the following identities, which hold in every group.

$$\begin{aligned}
[a,b]^{-1} &= [b,a].\\
[a,bc] &= [a,c][a,b]^c = [a,c][a,b][a,b,c].\\
[bc,a] &= [b,a]^c[c,a] = [b,a][b,a,c][c,a].\\
[a^{-1},b] &= [a,b]^{-a^{-1}}.\\
[a,b^{-1}] &= [a,b]^{-b^{-1}}.
\end{aligned}$$

We will use these without reference. The commutator of two subgroups A and B of a group, written $[A,B]$, is defined as the subgroup generated by commutators $[a,b]$, where $a \in A$ and $b \in B$. We will use the left normed convention here as well. Thus $[A_1,\dots,A_{m+1}] = [[A_1,\dots,A_m],A_{m+1}]$. From the first identity above we have that $[A,B]=[B,A]$. It is easy to calculate the following identity using the identities above.

$$[a,[b,c]] = [a,b^{-1}c^{-1}bc] = [a,b,c][a,c,b]^{-1}u,$$

where u is a product of commutators with entry set $\{a,b,c\}$ and weight at least 4. Note that we let a product of commutators include inverses of commutators. There is a similar identity for commutators of subgroups.

Lemma 2.4. (3 subgroups lemma.) *Let G be a group and $A,B,C \trianglelefteq G$. Then,*

$$[A,B,C] \leq [A,C,B][A,[B,C]].$$

Another useful identity, from which the 3 subgroups lemma follows, is the Hall-Witt identity

$$[a,b^{-1},c]^b[b,c^{-1},a]^c[c,a^{-1},b]^a = 1.$$

Now suppose that we have a commutator identity which holds for all elements in a group, say $f(x_1,\dots,x_m)=1$, where $f(x_1,\dots,x_m)$ is a product of commutators, each with entry set some subset of $\{x_1,\dots,x_m\}$. Suppose further that the commutators are ordered with respect to the entry set, such that a commutator with entry set X is to the left of a commutator with entry set $Y \neq X$ if $x_a \in X$, where $a = min\{1 \leq i \leq m : x_i \in (X \setminus Y) \cup (Y \setminus X)\}$. Setting $x_1 = 1$ we have that $f(1,x_2,\dots,x_m)=1$. Thus the product of commutators in f with no entry of x_1 is trivial, as is the product of commutators in f with at least one entry of x_1. Using this technique repeatedly, by setting $x_2=1$, then $x_3=1$ and so on, gives that the product of commutators in f with any particular entry set is trivial. We will use this idea throughout this thesis and at times without reference.

In many of the proofs for right n-Engel subgroups we will be concerned with showing that a commutator subgroup $[H,_i G]$ is trivial, where $H \trianglelefteq G$. The following lemma will be helpful when the group is finitely generated.

Lemma 2.5. *If $G = \langle x_1, \ldots, x_d \rangle$ and $H \trianglelefteq G$, then*

$$[H,_n G] = \langle [h, g_1, \ldots, g_n] : h \in H, g_i \in \{x_1, \ldots, x_d\} \rangle^G.$$

Proof. Since $H \trianglelefteq G$, it is clear that the right hand side is a subset of $[H,_n G]$. Thus it remains to prove the other inclusion. We prove this by induction on $n \geq 0$. The case $n = 0$ is trivial. Suppose that the case $n = k$ is true and let

$$K = \langle [h, g_1, \ldots, g_k] : h \in H, g_i \in \{x_1, \ldots, x_d\} \rangle^G.$$

Let $g \in [H,_{k+1} G]$, say $g = [k_1, f_1]^{\epsilon_1} \cdots [k_r, f_r]^{\epsilon_r}$, where $k_1, \ldots, k_r \in K$, $f_1, \ldots, f_r \in G$ and $\epsilon_1, \ldots, \epsilon_r \in \{-1, 1\}$. Using $[xy, f_i] = [x, f_i]^y [y, f_i]$ and $[x^{-1}, f_i] = [x, f_i]^{-x^{-1}}$ we may assume that each k_i is of the form $[h, g_1, \ldots, g_k]^f$, for some $f \in G$. So $[k_i, f_i] = [h, g_1, \ldots, g_k, f_i^{f^{-1}}]^f$. Now, each $f_i^{f^{-1}}$ is a product of the x_i's, so using $[k_i^{f^{-1}}, xy] = [k_i^{f^{-1}}, y][k_i^{f^{-1}}, x]^y$ and $[k_i^{f^{-1}}, x^{-1}] = [k_i^{f^{-1}}, x]^{-x^{-1}}$ we can write $[k_i, f_i]$ as a product of commutators in the desired form. $\square$

We will be working with nilpotent n-Engel groups. The analogue of being nilpotent for subgroups is that they lie in some term of the upper central series.

Definition 2.6. Let H be a subgroup of a group G. We say that H is *upper central (in G) with upper central degree i* if $[H,_i G] = \{1\}$ and $[H,_{i-1} G] \neq \{1\}$.

From the definition of a commutator we have the identity $ab = ba[a, b]$. Hence two consecutive elements in a product can be swapped, which adds in a commutator of the two elements. Thus for an arbitrary product of elements, these can be ordered as desired, which adds in more commutators. If the group is nilpotent, then the product can be written as a new product where commutators appear in ascending weights. If there is an ordering on the generators of the group, then the product can be written in ascending order with respect to this ordering, for each weight. This is known as Hall's collection process. A similar idea is behind the following theorem.

Theorem 2.7. (Hall-Petrescu formula.) *Let a, b be elements in a group and $f \in \mathbb{Z}$. Then,*

$$a^f b^f = (ab)^f w_2^{\binom{f}{2}} w_3^{\binom{f}{3}} \cdots w_f^{\binom{f}{f}},$$

where $w_i \in \gamma_i(\langle a, b \rangle)$, for $i = 1, 2, \ldots, f$.

Proof. For elements $a_1, \ldots, a_f, b_1, \ldots, b_f$ we have, by Hall's collection process,

$$a_1 \cdots a_f b_1 \cdots b_f = x_1 x_2 \cdots x_f \prod_{i_1 < i_2} x_{i_1, i_2} \cdots \prod_{i_1 < \ldots < i_f} x_{i_1, \ldots, i_f},$$

where $x_{i_1, \ldots, i_t} \in \gamma_t(\langle a_1, \ldots, a_f, b_1, \ldots, b_f \rangle)$ is a product of commutators with at least one entry from each set $\{a_{i_1}, b_{i_1}\}, \ldots, \{a_{i_t}, b_{i_t}\}$ and no other entries. For $i \in \{1, \ldots, f\}$, setting $a_j = b_j = 1 \ \forall j \neq i$ gives $a_i b_i = x_i$. By induction on t, setting $a_j = b_j = 1$ $\forall j \notin \{i_1, \ldots, i_t\}$ gives an expression for $x_{i_1, \ldots, i_t}$ in terms of $a_{i_1}, \ldots, a_{i_t}, b_{i_1}, \ldots, b_{i_t}$. Note that, by induction on t, the expression for $x_{i_1, \ldots, i_t}$ is the same as that for $x_{j_1, \ldots, j_t}$ with each a_{i_k} replaced with a_{j_k} and each b_{i_k} replaced with b_{j_k}. Letting $a_1 = \ldots = a_f = a$ and $b_1 = \ldots = b_f = b$, and letting w_t be $x_{1, \ldots, t}$ gives the result. $\square$

2.3 Basic commutators

When using free groups for counterexamples it will be useful to consider basic commutators. Suppose that $x_1, \ldots, x_r$ are generators of a group. Consider commutators with entry set $\{x_1, \ldots, x_r\}$. These can be ordered by weight and then given an arbitrary ordering within a specific weight. We can then define inductively by weight the basic commutators on this set with respect to this ordering.

Definition 2.8. (i) $x_1, \ldots, x_r$ are the *basic commutators of weight 1.*
(ii) The *basic commutators of weight* $m > 1$ are commutators of the form $[c_i, c_j]$, where c_i and c_j are basic commutators with combined weight m, $c_i > c_j$ and if $c_i = [c_k, c_l]$, then $c_l \leq c_j$.

The motivation behind this definition is that the basic commutators are exactly the commutators that can arise from Hall's collection process. The usefulness of basic commutators in terms of free groups comes from the following theorem, due to M. Hall [18].

Theorem 2.9. (Basis theorem.) *Let F be a free group on a finite set of generators and let m be a positive integer. Suppose that $c_1, \ldots, c_t$ are the basic commutators of weight at most m, with respect to some ordering. Then each $f \in F$ has a unique representation*

$$f = c_1^{e_1} c_2^{e_2} \cdots c_t^{e_t} \mod \gamma_{m+1}(F).$$

Also, the basic commutators of weight m form a basis for $\gamma_m(F)/\gamma_{m+1}(F)$.

For an example in Section 3.4 we will consider the free group F on generators x and y. We will quotient out by $\gamma_5(F)$ and be considering $\gamma_4(F)$. We can order the commutators in F with entry set $\{x, y\}$ so that in particular $x > y$. By the above theorem we need only consider the basic commutators of weight 4, and there is no relation between these. From the definition one sees that in this case the basic commutators of weight 4 are $[x, y, y, y]$, $[x, y, y, x]$ and $[x, y, x, x]$.

In sections 6.3 and 6.5 we will construct groups starting with a free group on a set of generators $\{h, x_1, \ldots, x_m\}$, for some m. We will be interested in left normed commutators of a particular weight, say t, with entry set $\{h, x_1, \ldots, x_m\}$, first entry h and no other entry of h. Let the set of these commutators be X. It will be important that there are no relations between these commutators in the free group modulo commutators of higher weight. Consider an arbitrary ordering on commutators with entry set $\{h, x_1, \ldots, x_m\}$ and let Y be the set of basic commutators of weight t with exactly one h entry. Lemma 11.2.1 from [19] shows that there is a bijection from X to Y. Also, X is contained in the span of Y modulo higher weight commutators by Theorem 2.9. For any commutator in Y, one can use the commutator identity $[a, b]^{-1} = [b, a]$ to move h first, and then use $[a, [b, c]] = [a, b, c][a, c, b]^{-1}$ modulo higher weight commutators to see that the commutator in Y is in the span of X modulo higher weight commutators. Thus the spans of X and Y are the same modulo higher weight commutators and $|X| = |Y|$, so the elements of X are independent modulo higher weight commutators.

2.4 Lie rings

The solutions to both the restricted Burnside and the restricted local nilpotence problems use Lie rings. These will also be used for some of the proofs in Chapter 4.

Definition 2.10. A *Lie ring* is a triple $(L,+,\cdot)$, where
(i) $(L,+)$ is an abelian group.
(ii) $u \cdot u = 0$, $\forall u \in L$.
(iii) $(u \cdot v) \cdot w + (v \cdot w) \cdot u + (w \cdot u) \cdot v = 0$, $\forall u, v, w \in L$.
(iv) $(u + v) \cdot w = u \cdot w + v \cdot w$, $\forall u, v, w \in L$.

Condition (iii) is known as the Jacobi identity. We use the standard left normed convention for Lie products. Thus $a_1 a_2 \cdots a_{n+1} = (a_1 \cdots a_n) \cdot a_{n+1}$. Lie rings are a useful tool for dealing with Engel groups via the associated Lie ring of a group. For this consider the lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots,$$

where $\gamma_{i+1}(G) = [\gamma_i(G), G]$. Let $L_i(G) = G_i/G_{i+1}$ and consider as an abelian group

$$L(G) = L_1(G) \oplus L_2(G) \oplus \cdots .$$

We define a multiplication on $L(G)$ by letting $aG_{i+1} \cdot bG_{j+1} = [a,b]G_{i+j+1}$ for $a \in G_i$ and $b \in G_j$. This is then extended linearly to the whole of $L(G)$. To see that this is well-defined we will use that $[G_i, G_j] \leq G_{i+j}$, which we prove by induction on $i \geq 1$. The induction basis is trivial as $[G, G_j] = [G_j, G] = G_{j+1}$. Suppose that $[G_i, G_j] \leq G_{i+j}$ for some i, then

$$\begin{aligned}
[G_{i+1}, G_j] =& [G_i, G, G_j] \\
\leq& [G_i, G_j, G][G_i, [G, G_j]], \text{ by the 3 subgroups lemma (Lemma 2.4)}, \\
\leq& [G_{i+j}, G][G_i, G_{j+1}] \\
\leq& G_{i+j+1}.
\end{aligned}$$

This completes the induction. Now suppose that $a, c \in G_i$, $b, d \in G_j$, $a = cu$, where $u \in G_{j+1}$ and $b = dv$, where $v \in G_{j+1}$. Then

$$\begin{aligned}
[a,b]G_{i+j+1} &= [cu, dv]G_{i+j+1} \\
&= [cu, v][cu, d][cu, d, v]G_{i+j+1} \\
&= [cu, d]G_{i+j+1} \\
&= [c, d][c, d, u][u, d]G_{i+j+1} \\
&= [c, d]G_{i+j+1}.
\end{aligned}$$

Thus the multiplication is well-defined. To see that $L(G)$ is a Lie ring we need to check conditions (ii) - (iv) in the definition. These follow from some of the identities in Section 2.2. Condition (ii) follows from $[a,a] = 1$ and $[a,b] = [b,a]^{-1}$, condition (iv) from $[bc, a] = [b,a]^c[c,a]$ and the Jacobi identity follows from the Hall-Witt identity. For any group G we now have an associated Lie ring $L(G)$.

An element a in a Lie ring L is right n-Engel if $ax^n = 0$ for all $x \in L$ and L is n-Engel if all $a \in L$ are right n-Engel. When G is an n-Engel group we can deduce certain properties of $L(G)$. First let $x, y_1, \ldots, y_n \in G$. Then

$$[x, \underbrace{y_1 \cdots y_n, \ldots, y_1 \cdots y_n}_{n \text{ times}}] = 1.$$

Expanding this using commutator identities gives

$$(\prod_{\sigma\in S_n}[x,y_{\sigma(1)},\ldots,y_{\sigma(n)}])z=1,$$

where z is a product of commutators with entry set $\{x,y_1,\ldots,y_n\}$ and weight at least $n+2$. Now let $u=aG_{i+1}\in L_i$, $v_1=b_1G_{j_1+1}\in L_{j_1}$, $\ldots$, $v_n=b_nG_{j_n+1}\in L_{j_n}$. Then

$$\begin{aligned}\sum_{\sigma\in S_n}uv_{\sigma_1}\cdots v_{\sigma_n} &= \prod_{\sigma\in S_n}[a,b_{\sigma(1)},\ldots,b_{\sigma(n)}]G_{i+j_1+\cdots+j_n+1}\\ &= 1G_{i+j_1+\cdots+j_n+1}\\ &= 0.\end{aligned}$$

By the multilinearity of the multiplication in $L(G)$, we have that $L(G)$ satisfies the linearised n-Engel identity

$$\sum_{\sigma\in S_n}xy_{\sigma_1}\cdots y_{\sigma_n}=0.$$

Note that setting $y_1=\ldots=y_n=y$ gives $n!(xy^n)=0$. Thus if the characteristic of $L(G)$ is not divisible by any prime $p\le n$, then $L(G)$ is an n-Engel Lie ring. In general $L(G)$ need not be n-Engel. However, if $u=aG_{i+1}\in L_i$ and $v=bG_{j+1}\in L_j$, then

$$uv^n=[a,\underbrace{b,\ldots,b}_{n\text{ times}}]G_{i+nj+1}=1G_{i+nj+1}=0.$$

By multilinearity we have that $L(G)$ satisfies $xv^n=0$, for any $x\in L(G)$ and $v\in L_j$ for some $j\ge 1$.

These properties of $L(G)$ become very useful when considering the following two theorems of Zel'manov [39, 41, 42].

Theorem 2.11. *Let $L=\langle x_1,\ldots,x_d\rangle$ be a d-generator Lie ring and let n,m be positive integers such that the following two properties hold.*
(1) $\sum_{\sigma\in S_n}uv_{\sigma(1)}\cdots v_{\sigma(n)}=0$ for all $u,v_1,\ldots,v_n\in L$,
(2) $uv^m=0$ for all $u\in L$ and all Lie products v in the generators.
Then L is nilpotent of (d,n,m)-bounded class.

Theorem 2.12. *Let L be a torsion-free n-Engel Lie ring. Then L is nilpotent of n-bounded class.*

These theorems have had a big influence on the study of Engel groups. The first theorem was used by Zel'manov to give a solution to the restricted Burnside problem in the special case of groups of prime power exponent. It also gives the solution to the restricted local nilpotence problem via the theorem of Wilson [38].

In order to apply these theorems later on we will make use of semidirect products of Lie rings. In order to define these we first need to introduce Lie ring homomorphisms, derivations and actions. For Lie rings L and M, a map $\phi : L \mapsto M$ is a *Lie ring homomorphism* if it preserves addition and multiplication. That is, for $x_1, x_2 \in L$, $\phi(x_1 + x_2) = \phi(x_1) + \phi(x_2)$ and $\phi(x_1 \cdot x_2) = \phi(x_1) \cdot \phi(x_2)$. A *derivation* of a Lie ring L is a map $D : L \mapsto L$ such that, for every $x_1, x_2 \in L$, $D(x_1x_2) = D(x_1)x_2 + x_1D(x_2)$. The set of derivations of L is denoted $\mathrm{Der}L$. This is a Lie ring when equipped with the product $(D_1 \cdot D_2)(x) = D_1(x)D_2(x) - D_2(x)D_1(x)$. If L and M are Lie rings, then we say that M *acts on* L *as a derivation* if there is a Lie ring homomorphism from M to $\mathrm{Der}L$.

Definition 2.13. If M acts on L as a derivation via $\theta : M \mapsto \mathrm{Der}L$, then the *semidirect product of* L *and* M *with respect to* θ, written $L \rtimes_\theta M$, is the abelian group $L \oplus M$ with multiplication defined by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + \theta(y_2)x_1 - \theta(y_1)x_2, y_1y_2).$$

One can check that the semidirect product is itself a Lie ring. When the map θ is clear we will write $L \rtimes M$ for the semidirect product. If, for $x \in L$ and $y \in M$, one uses the notation xy for $\theta(y)x$ and yx for $-\theta(y)x$, then the multiplication in the semidirect product becomes

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 + x_1y_2 + y_1x_2, y_1y_2),$$

or written differently

$$(x_1 + y_1) \cdot (x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2.$$

We will use this latter notation.

2.5 Nilpotent groups and isolators

Here we discuss some properties of nilpotent groups and isolators. In Chapter 3 we will be dealing with nilpotent groups, but in fact we will see that this can be relaxed to locally nilpotent or residually nilpotent groups, which are defined as follows.

Definition 2.14. A group G is said to be *residually nilpotent* if $\cap_{i=1}^{\infty}\gamma_i(G) = \{1\}$.

In Section 3.2 we will state four theorems on n-Engel groups, which we will generalise in Chapter 4. It turns out that one of these theorems follows from two of the others and to show this the following lemma will be used.

Lemma 2.15. *Let H be an upper central normal subgroup of a group G and let $\phi_i : G \mapsto G/Z_i(G)$ be the natural homomorphism, i.e. $\phi_i(g) = gZ_i(G)$. If H is torsion-free, then so is $\phi_i(H)$.*

Proof. Assume that H is torsion-free and has upper central degree c in G. We show by induction on $i \geq 0$ that $\phi_i(H)$ is torsion-free. Since $Z_0(G) = \{1\}$, the case $i = 0$ is trivial. Let $k \in \mathbb{N}$ and suppose that the statement is true for all $i < k$. Suppose that $h^l \in Z_k(G)$, for some $h \in H$. It remains to see that $h \in Z_k(G)$. We show that, for $g_1, \ldots, g_m \in G$, $[h, g_1, g_2, \ldots, g_m] \in Z_{k-1}(G)$, by reverse induction on $m \geq 1$. The case $m \geq c$ is trivial. Suppose that this holds for values greater than m. Now, $[h^l, g_1, \ldots, g_m] \in Z_{k-m}(G) \leq Z_{k-1}(G)$ and so by inductive hypothesis for the reverse induction, $[h, g_1, \ldots, g_m]^l \in Z_{k-1}(G)$. Thus, as $\phi_{k-1}(H)$ is torsion-free by inductive hypothesis, we have $[h, g_1, \ldots, g_m] \in Z_{k-1}(G)$, for all $m \geq 1$. In particular $[h, g_1] \in Z_{k-1}(G)$ for all $g_1 \in G$ and so $h \in Z_k(G)$. $\square$

For the proof of Theorem 4.2 we will be using isolators. We will need to use some basic properties of isolators, which we prove here. First the definition.

Definition 2.16. Let K be a subgroup of a group F. The *isolator of K in F* is $\sqrt[F]{K} = \{t \in F :$ there exists a positive integer r such that $t^r \in K\}$.

This definition will only be useful to us if the isolator is a subgroup. When the group is locally nilpotent this is the case, as we now show.

Lemma 2.17. *Let F be a locally nilpotent group with subgroup K. Then $\sqrt[F]{K}$ is a subgroup of F.*

Proof. Suppose that $x, y \in F$ and that $x^i, y^j \in K$, for some $i, j \in \mathbb{N}$. Since K is a subgroup of F, $1 \in \sqrt[F]{K}$ and $x^{-i} \in K$. It remains only to see that $xy \in \sqrt[F]{K}$. Let $H = \langle x, y\rangle$ have nilpotency class c. Now, for $m \geq 1$ and $k_1, \ldots, k_m \in \{x, y\}$ we have, modulo $\gamma_{m+1}(H)$,

$$[k_1, \ldots, k_m]^{(ij)^m} = [k_1^{ij}, \ldots, k_m^{ij}] \in K \cap H.$$

Modulo $\gamma_{m+1}(H)$, the commutators $[k_1, \ldots, k_m]$ generate $\gamma_m(H)$ and commute. Thus $\gamma_m(H)^{(ij)^m} \leq \gamma_{m+1}(H)(K \cap H)$ for all $m \geq 1$. Let $l = 1 + 2 + \cdots + c$. Then $(xy)^{(ij)^l} \in H^{(ij)^l} \leq \gamma_{c+1}(H)(K \cap H) = K \cap H \leq K$. Hence $xy \in \sqrt[F]{K}$. $\square$

Note that if $\sqrt[F]{K}$ is a subgroup of F and K is normal in F, then $\sqrt[F]{K}$ is also normal in F, since if $x, g \in G$ and $x^i \in K$, then $(g^{-1}xg)^i = g^{-1}x^i g \in K$. Also, we get that $F/\sqrt[F]{K}$ is torsion-free, since if $f^l \in \sqrt[F]{K}$, then $f \in \sqrt[F]{K}$. Taking $K = \{1\}$ in Lemma 2.17 gives the result that the set of torsion elements of a locally nilpotent group form a normal subgroup. We now give a different setting in which the isolator is a subgroup.

Lemma 2.18. *Suppose that $N \trianglelefteq F$ with F/N locally nilpotent and $N \leq K \leq F$. Then $\sqrt[F]{K}$ is a subgroup of F.*

Proof. The identity element is trivially in $\sqrt[F]{K}$ and this is clearly closed under taking inverses. It remains to see that it is closed under taking products. Let $g, h \in \sqrt[F]{K}$. Then $gN, hN \in \sqrt[F/N]{K/N}$. Since F/N is locally nilpotent, we have, by Lemma 2.17, that $\sqrt[F/N]{K/N}$ is a subgroup of F/N. Thus $ghN \in \sqrt[F/N]{K/N}$ and so $(gh)^l N = (ghN)^l \in K/N$ for some $l \in \mathbb{N}$. So $(gh)^l \in K$ and hence $gh \in \sqrt[F]{K}$. $\square$

2.6 Powerful p-groups

In this thesis we are interested in the structure of nilpotent n-Engel groups and upper central right n-Engel subgroups. A particular class of nilpotent groups is locally finite p-groups. We will see in Section 3.4 that there are finite n-Engel p-groups of arbitrary nilpotency class for fixed n and p. There is however a bound on the nilpotency class for n-Engel powerful p-groups, for fixed n and p, as was shown by Abdollahi and Traustason [1]. We will generalise this result for right n-Engel subgroups that are powerfully embedded in some finite p-group. We will also generalise some consequences of this theorem. Here we give an overview of powerful p-groups, stating some basic properties which will be used later. The details can be found in [12].

Definition 2.19. A finite p-group G is *powerful* if p is odd and $[G, G] \leq G^p$ (i.e. G/G^p is abelian), or if $p = 2$ and $[G, G] \leq G^4$ (i.e. G/G^4 is abelian).

Definition 2.20. A subgroup H of a finite p-group G is *powerfully embedded* in G, written H *p.e.* G, if p is odd and $[H, G] \leq H^p$, or if $p = 2$ and $[H, G] \leq H^4$.

From the definition one can see that powerful p-groups are generalised abelian groups. It turns out that they have similar properties to abelian groups. Notice that if H is powerfully embedded in G, then H is powerful. It also follows from the definition of powerfully embedded that for H *p.e.* G and $N \trianglelefteq G$, HN/N *p.e.* G/N. We will need the following well known properties of powerful groups and powerfully embedded subgroups. Firstly if H *p.e.* G, then H^p *p.e.* G. Secondly, just as with abelian groups, if G is a powerful p-group, then $G^p = \{g^p : g \in G\}$. From these properties we can prove by induction on $k \geq 1$ that for every powerful p-group G, $G^{p^k} = \{g^{p^k} : g \in G\}$. The induction basis is mentioned above. If the claim is true for some k, then $G^{p^{k+1}} \leq (G^{p^k})^p = \{g^p : g \in G^{p^k}\} = \{g^{p^{k+1}} : g \in G\}$. From this we have that if H is powerfully embedded in G, then $H^{p^k} = \{h^{p^k} : h \in H\}$ and H^{p^k} is powerfully embedded in G, for any positive integer k.

Chapter 3

n-Engel Groups

3.1 Introduction

In this chapter we consider some theorems on the structure of nilpotent n-Engel groups, which we shall generalise to theorems on upper central right n-Engel subgroups in Chapter 4. In particular, one of these theorems is new, for which we shall give a proof. These theorems are either about bounding the nilpotency class or are closely related. This gives rise to certain integer valued functions that describe the best possible upper bounds. We will find the values of these functions for $n \leq 3$ and state what is known for $n = 4$. Much of the work appears in [9], with more values added here and a slightly altered version of the main result and proof.

We then look at locally finite n-Engel p-groups and again state theorems to be generalised in Chapter 4. Again these theorems are either about bounding the nilpotency class or are closely related and give rise to certain integer valued functions describing the best upper bounds. We give the best possible values for $n \leq 2$ and also find some upper bounds for $n = 3$.

3.2 n-Engel group theorems

The main open question about n-Engel groups is whether they are locally nilpotent, that is whether finitely generated n-Engel groups are nilpotent. The first theorem we state, due to Wilson [38], shows that the n-Engel groups of a particular finite rank that are nilpotent have nilpotency class bounded in terms of n and the rank.

Theorem 3.1. *Let G be a d-generator nilpotent n-Engel group. Then G is nilpotent of class bounded by a function in d and n.*

A theorem of Zel'manov [40] shows that the situation is similar for torsion-free groups.

Theorem 3.2. *Let G be a torsion-free nilpotent n-Engel group. Then G is nilpotent of class bounded by a function of n.*

We will use these two theorems to prove that, for any particular n, every nilpotent n-Engel group raised to some exponent, dependent only on n, is contained in a term of the upper central series, again dependent only on n.

Theorem 3.3. *([9]) Let G be a nilpotent n-Engel group. There exist positive integers $c = c(n)$ and $e = e(n)$, only dependent on n, such that $G^e \leq Z_c(G)$.*

This theorem was preceded by a result of Burns and Medvedev [6], who proved under the same assumptions that there exist positive integers $c = c(n)$ and $e = e(n)$, only dependent on n, such that G^e is nilpotent of class c. Another similar result of Burns and Medvedev is the following theorem [6].

Theorem 3.4. *Let G be a nilpotent n-Engel group. There exist positive integers $c = c(n)$ and $e = e(n)$, only dependent on n, such that $\gamma_c(G)^e = \{1\}$.*

Note that Theorem 3.4 implies Theorem 3.2. Also, let G be a torsion-free nilpotent n-Engel group. By Theorem 3.3 we can let $c = c(n)$ and $e = e(n)$ be such that $G^e \leq Z_c(G)$. By Lemma 2.15, $G/Z_c(G)$ is torsion-free. Thus, $G \leq Z_c(G)$ is nilpotent of class at most c. Thus Theorem 3.3 also implies Theorem 3.2. We will, however, use Theorem 3.2 to prove Theorem 3.3.

We can define integer valued functions to describe the best upper bounds for values arising from the theorems.

Definition 3.5. Let d and n be positive integers.
(i) Define $c_1(d, n)$ to be the smallest positive integer such that every d-generator nilpotent n-Engel group has nilpotency class at most $c_1(d, n)$.
(ii) Define $c_2(n)$ to be the smallest positive integer such that every torsion-free nilpotent n-Engel group has nilpotency class at most $c_2(n)$.
(iii) Define $c_3(n)$ to be the smallest positive integer such that every nilpotent n-Engel group G satisfies $G^{e(n)} \leq Z_{c_3(n)}(G)$ for some positive integer $e(n)$ dependent only on n. Then define $f_3(n)$ to be the smallest possible value of $e(n)$ here. Define $e_3(n)$ to be the smallest positive integer such that every nilpotent n-Engel group G satisfies $G^{e_3(n)} \leq Z_{c(n)}(G)$ for some positive integer $c(n)$ dependent only on n. Then define $d_3(n)$ to be the smallest possible value of $c(n)$ here.

(iv) Define $c_4(n)$ to be the smallest positive integer such that every nilpotent n-Engel group G satisfies $\gamma_{c_4(n)}(G)^{e(n)} = \{1\}$ for some positive integer $e(n)$ dependent only on n. Then define $f_4(n)$ to be the smallest possible value of $e(n)$ here. Define $e_4(n)$ to be the smallest positive integer such that every nilpotent n-Engel group G satisfies $\gamma_{c(n)}(G)^{e_4(n)} = \{1\}$ for some positive integer $c(n)$ dependent only on n. Define $d_4(n)$ to be the smallest possible value of $c(n)$ here.

The statements of Theorems 3.1 - 3.4 were originally to do with locally nilpotent or residually nilpotent n-Engel groups. In fact in each of the theorems the condition that the group is nilpotent can be relaxed to the group being either locally nilpotent or residually nilpotent. Whilst these conditions are weaker than nilpotence, in this setting they are equivalent as the following remark shows.

Remark 3.6. Theorems 3.1 - 3.4 are equivalent to the theorems with the assumption that G is nilpotent replaced by the condition G is locally nilpotent or residually nilpotent.

Proof. Clearly a nilpotent group is locally nilpotent and residually nilpotent. Suppose that G is locally nilpotent instead of nilpotent. Theorem 3.1 still holds as G is finitely generated and hence nilpotent. For Theorem 3.2 consider an element in $\gamma_{c_2(n)}(G)$. This can be written as a finite product of commutators of weight at least $c_2(n)$. The subgroup generated by the entries of these commutators is then nilpotent and hence the theorem applies and the element is trivial. As the element was arbitrary the theorem holds. For Theorem 3.3 consider an element in $[G^{f_3(n)},_{c_3(n)} G]$. This can be written as a finite product u of commutators of weight at least $c_3(n)+1$ with first entry in $G^{f_3(n)}$. Each element in $G^{f_3(n)}$ can be written as a finite product of elements of the form $g^{f_3(n)}$. The subgroup generated by each such g appearing in u and the other entries of the commutators is nilpotent and thus the theorem applies and $u = 1$. For Theorem 3.4 consider an element in $\gamma_{c_4(n)}(G)$. This can be written as a finite product of commutators of weight at least $c_4(n)$. The subgroup generated by the entries of these commutators is then nilpotent and hence the theorem applies and the element raised to the power $f_4(n)$ is trivial. Hence $\gamma_{c_4(n)}(G)^{f_4(n)} = \{1\}$.

Now suppose that G is residually nilpotent instead of nilpotent. For arbitrary $i \in \mathbb{N}$, $G/\gamma_i(G)$ is a nilpotent n-Engel group. Thus applying Theorem 3.1 to this gives that $\gamma_{c_1(n)+1}(G) \leq \gamma_i(G)$, for all $i \in \mathbb{N}$. Thus $\gamma_{c_1(n)+1}(G) \leq \cap_{i=1}^{\infty}\gamma_i(G) = \{1\}$. Similarly, applying Theorem 3.3 we get $[G^{f_3(n)},_{c_3(n)} G] \leq \cap_{i=1}^{\infty}\gamma_i(G) = \{1\}$ and applying Theorem 3.4 we get $\gamma_{c_4(n)}(G)^{f_4(n)} \leq \cap_{i=1}^{\infty}\gamma_i(G) = \{1\}$. For Theorem 3.2 consider a residually nilpotent torsion-free n-Engel group G. Applying Theorem 3.4 we have

$\gamma_{c_4(n)}(G)^{f_4(n)} = \{1\}$ and as G is torsion-free we have $\gamma_{c_4(n)}(G) = \{1\}$. □

From the proof of this remark one sees that the functions arising from the theorems would be unchanged if we replaced nilpotent with either locally nilpotent or residually nilpotent.

By Theorem 2.3 we can state Theorem 3.1 in two more equivalent ways. One is that every finitely generated residually nilpotent n-Engel group is nilpotent. The other is that the locally nilpotent n-Engel groups form a subvariety of the variety of n-Engel groups. In particular, from this it makes sense to talk about the relatively free locally nilpotent n-Engel group on a set of generators.

3.3 Proof of Theorem 3.3

In this section we will use Theorem 3.1 and Theorem 3.2 to prove Theorem 3.3. It will be easily derived from the following Lemma.

Lemma 3.7. *Let G be a nilpotent n-Engel group. Then there exist positive integers $l = l(n)$ and $k = k(n)$, only dependent on n, such that*

$$[g, g_1, \ldots, g_k]^l = 1$$

for all $g, g_1, \ldots, g_k \in G$.

Proof. Let $k = k(n) = c_2(n)$ and let $F = \langle x, x_1, \ldots, x_k \rangle$ be the relatively free n-Engel group on $k+1$ generators that is nilpotent of class $c_1(k+1, n)$, where $c_1(k+1, n)$ and $c_2(n)$ are as defined in Definition 3.5. We will use the fact that the set of torsion elements in a nilpotent group form a normal subgroup, which follows from Lemma 2.17. Thus we may quotient out by the torsion subgroup of F, say T. Since F/T is a torsion-free nilpotent n-Engel group, by Theorem 3.2 and the definition of k we have that $\gamma_{k+1}(F/T) = \{1\}$. Thus $\gamma_{k+1}(F) \leq T$. Then we have

$$[x, x_1, \ldots, x_k]^l = 1$$

for some $l = l(n)$. Let G be a nilpotent n-Engel group and suppose that $K = \langle g, g_1, \ldots, g_k \rangle \leq G$. By Theorem 3.1 and the definition of $c_1(k+1, n)$, K is nilpotent of class at most $c_1(k+1, n)$. Hence K is a quotient of F and we may consider a homomorphism ϕ from F to K that maps x to g and x_i to g_i, for each i. Then $[g, g_1, \ldots, g_k]^l = \phi([x, x_1, \ldots, x_k]^l) = 1$. This proves the Lemma. □

We now derive Theorem 3.3 from this Lemma. Let G be a nilpotent n-Engel group and $k = k(n)$ and $l = l(n)$ be as in Lemma 3.7. Let $g, g_1, \ldots, g_k \in G$ and $K = \langle g, g_1, \ldots, g_k \rangle$. By Theorem 3.1, K is nilpotent of n-bounded class $c = c(n) = c_1(k(n)+1, n)$. We prove by reverse induction on $0 \le r \le c-k$ that the law

$$[x^{l^{c-k-r}}, x_1, \ldots, x_k, h_1, \ldots, h_r] = 1$$

holds in K. Since K is nilpotent of class at most c, the case $r = c-k$ holds. Now suppose that $0 \le r \le c-k-1$ and that the result holds for larger values of r in $\{0, 1, \ldots, c-k\}$. By the induction hypothesis we have that $x^{l^{c-k-r-1}}$ is in the $(k+r+1)$th centre. Hence

$$\begin{aligned}[x^{l^{c-k-r}}, x_1, \ldots, x_k, y_1, \ldots, y_r] &= [(x^{l^{c-k-r-1}})^l, x_1, \ldots, x_k, y_1, \ldots, y_r] \\ &= [x^{l^{c-k-r-1}}, x_1, \ldots, x_k, y_1, \ldots, y_r]^l.\end{aligned}$$

By Lemma 3.7 this is trivial, which proves the induction step. Taking $r = 0$ we get

$$[x^{l^{c-k}}, x_1, \ldots, x_k] = 1$$

and so the theorem is proved.

3.4 Examples for small n

In this section we find the values of the functions c_i, d_i, e_i and f_i from Definition 3.5 for small n. We will only be concerned with $n \le 4$. For these values of n we know that n-Engel groups are locally nilpotent. Thus, by Remark 3.6, we can remove nilpotent from the definition of these functions for these n. Thus, for example, $c_1(d, n)$ is the smallest positive integer such that every d-generator n-Engel group is nilpotent of class at most $c_1(d, n)$, for $n \le 4$. We first state some remarks about these functions, some of which will help in finding particular values.

Remark 3.8. (i) Since n-Engel groups are also m-Engel groups for each $m > n$, each $c_1(d, n)$, $c_i(n)$, $d_i(n)$, $e_i(n)$ and $f_i(n)$ is an increasing function in n.
(ii) If G is a torsionfree nilpotent n-Engel group, then by Theorem 3.4 G is nilpotent of class at most $c_4(n) - 1$. Further, as $G/Z_{c_3(n)}(G)$ is torsion-free, by Theorem 3.3 G is nilpotent of class at most $c_3(n)$. From the proof of Theorem 3.3 we see that $c_2(n) = c_3(n)$. It can also be seen from the proof of Theorem 3.4 (see [6]) that $c_2(n) = c_4(n) - 1$.
(iii) Let p be a prime and $f_3(n) = p^r m$, where $r > 0$ and p does not divide m. Then there is a nilpotent n-Engel group G such that $[G^m, {}_{c_3(n)} G] \ne \{1\}$. Finitely gener-

ated nilpotent groups are residually finite. Hence there is a finite nilpotent n-Engel group G such that $[G^m,_{c_3(n)} G] \neq \{1\}$. Finite nilpotent groups are the direct product of their Sylow subgroups and for each prime q, quotienting out by each Sylow q'-subgroup for primes $q' \neq q$, we have that there is a finite n-Engel p'-group G such that $[G^m,_{c_3(n)} G] \neq \{1\}$. It follows from the definition of $f_3(n)$ that $p' = p$ and hence there is a finite n-Engel p-group with nilpotency class greater than $c_3(n)$. Similarly for $f_4(n)$. Hence for a particular n the prime divisors of both $f_3(n)$ and $f_4(n)$ are the same, since these are the primes p where there exists a finite n-Engel p-group with nilpotency class greater than $c_3(n)$.
(iv) For a particular n, the prime divisors of both $e_3(n)$ and $e_4(n)$ are the same, since these are the primes p where there is no upper bound on the nilpotency class of finite n-Engel p-groups.
(v) $e_3(n)$ and $e_4(n)$ divide $f_3(n)$ and $f_4(n)$ respectively.
(vi) Let $F_{n,d}$ be the relatively free locally nilpotent n-Engel group on a set of d generators and $F_{n,\infty}$ be the relatively free locally nilpotent n-Engel group on a set of countably infinitely many generators, with torsion subgroup T_n. Then $c_1(d,n)$ is the nilpotency class of $F_{n,d}$, $c_2(n)$ is the nilpotency class of $F_{n,\infty}/T_n$ and the values of $c_3(n)$, $c_4(n)$, $d_3(n)$, $d_4(n)$, $e_3(n)$, $e_4(n)$, $f_3(n)$ and $f_4(n)$ are the corresponding values for $F_{n,\infty}$.

We now turn to finding these values for small n. The case $n = 1$ is trivial, since the 1-Engel groups are exactly the abelian groups. Thus, for any positive integer d, $c_1(d,1) = c_2(1) = c_3(1) = f_3(1) = d_3(1) = e_3(1) = f_4(1) = e_4(1) = 1$ and $c_4(1) = d_4(1) = 2$.

We now consider the first non-trivial case of 2-Engel groups. It is well known (see for example [34]) that we get the identities

$$\begin{aligned}
[x, x_1, x_2] &= [x, x_2, x_1]^{-1}, \\
[x, x_1, x_2]^3 &= 1, \\
[x^3, x_1, x_2] &= 1, \\
[x, x_1, x_2, x_3] &= 1.
\end{aligned}$$

Clearly $c_1(1,n) = 1$ for any n. From the first identity we see that $c_1(2,2) \leq 2$ and from the fourth we have $c_1(d,2) \leq 3$ for $d > 2$. The Burnside groups $B(2,3)$ and $B(3,3)$, that is the relatively free groups of rank 2 and 3 respectively and exponent 3, are 2-Engel, since they are of exponent three. They are known to be nilpotent of class 2 and 3 respectively and hence $c_1(2,2) = 2$ and $c_1(d,2) = 3$ for $d > 2$. The fourth identity gives $e_3(2) = e_4(2) = 1$. The group $B(3,3)$ then shows that $d_3(2) = 3$ and $d_4(2) = 4$.

Consider the relatively free 2-Engel group on two generators, x and y. Since this is the relatively free group on two generators of class two (as $c_1(2,2) = 2$), we have that $[x^l, y] = [x, y]^l \neq 1$ for every positive integer l. Thus we see that $c_2(2) = c_3(2) = 2$ and $c_4(2) = 3$. The third identity shows that $f_3(2) = 3$ and the second identity, along with the fourth identity, shows that $f_4(2) = 3$.

We now describe the case $n = 3$. Kappe and Kappe [25] have shown that all commutators in a 3-Engel group with a triple entry, that is three identical entries, are trivial. Hence $c_1(d, 3) \leq 2d$. Gupta and Newman [17] have shown that for $d > 2$, $c_1(d, 3) \leq 2d - 1$. From an example of Bachmuth and Mochizuki [3] we have equality here, that is for $d > 2$, $c_1(d, 3) = 2d - 1$. It is trivial that $c_1(1, 3) = 1$, so we would like to find $c_1(2, 3)$. We already have $c_1(2, 3) \leq 4$ and we will see that there is equality here. Consider the relatively free nilpotent group of class 4 on two generators, x and y. Quotient out by the normal closure of $[x, y, y, y]$, $[x, y, x, x]$ and $[x, y, y, x]^2$ and call the resulting group F. Note that $1 = [x, y, [x, y]] = [x, y, x, y][x, y, y, x]^{-1} = [x, y, x, y][x, y, y, x]$ in F. Now let f_1 and f_2 be arbitrary elements of F. We may write these in the forms $f_1 = x^{i_1}y^{j_1}u$ and $f_2 = x^{i_2}y^{j_2}v$, where $u, v \in F'$. Then,

$$\begin{aligned}
[f_1, f_2, f_2, f_2] =&[x^{i_1}y^{j_1}, x^{i_2}y^{j_2}, x^{i_2}y^{j_2}, x^{i_2}y^{j_2}]\\
=&[x, y, x, x]^{i_1 i_2^2 j_2}([x, y, y, x][x, y, x, y])^{i_1 i_2 j_2^2}\\
&[y, x, y, y]^{j_1 i_2 j_2^2}([y, x, x, y][y, x, y, x])^{j_1 i_2^2 j_2}\\
=&1.
\end{aligned}$$

Hence F is a 3-Engel group. But in this group $[x, y, y, x] \neq 1$, as there are no further relations between the basic commutators $[x, y, y, y]$, $[x, y, x, x]$ and $[x, y, y, x]$, by Theorem 2.9. Hence we have $c_1(2, 3) = 4$.

Heineken [22] proved that $c_4(3) \leq 5$ and that the only possible prime divisors of $e_3(3)$, $e_4(3)$, $f_3(3)$ and $f_4(3)$ are 2 and 5. An example of Kappe and Kappe ([25], example 1) of a torsion-free 3-Engel group of class 4 shows that $c_4(3) = 5$ and $c_2(3) = c_3(3) = 4$. An example to show that the prime 2 divides $e_3(3)$, $e_4(3)$, $f_3(3)$ and $f_4(3)$ is $C_2 \operatorname{wr} C_2^t$, for arbitrary $t \geq 1$, which we look at later in this section. Bachmuth and Mochizuki [3] have shown that the prime 5 divides $e_3(3)$, $e_4(3)$, $f_3(3)$ and $f_4(3)$, by finding an insoluble 3-Engel group of exponent 5. Gupta and Newman [17] have proved that $f_4(3) = 20$. They show that in any 3-Engel group we have the identities

$$[x, x_1, x_2, x_3, x_4]^{20} = 1,$$
$$[x, x_1, x_2, x_3, x_4, x_5]^{10} = 1.$$

We will now see that $f_3(3) = e_3(3) = 20$ and hence $d_3(3) = 4$. First we show that every 3-Engel group satisfies the law

$$[x^{20}, x_1, x_2, x_3, x_4] = 1.$$

Using the fact that all commutators with a triple entry are trivial we have that $[y^{20}, z] = [y, z]^{20}[y, z, y]^{\binom{20}{2}}$ for all y, z. Using this fact and the two identities of Gupta and Newman above, we have

$$\begin{aligned}
[x^{20}, x_1, x_2, x_3, x_4] =&[[x, x_1]^{20}[x, x_1, x]^{190}, x_2, x_3, x_4]\\
=&[[x, x_1]^{20}, x_2, x_3, x_4][[x, x_1, x]^{190}], x_2, x_3, x_4]\\
=&[[x, x_1, x_2]^{20}[x, x_1, x_2, [x, x_1]]^{190}, x_3, x_4][x, x_1, x, x_2, x_3, x_4]^{190}\\
=&[[x, x_1, x_2]^{20}, x_3, x_4][x, x_1, x_2, [x, x_1], x_3, x_4]^{190}\\
=&[[x, x_1, x_2, x_3]^{20}, x_4][x, x_1, x_2, x_3, [x, x_1, x_2], x_4]^{190}\\
=&[x, x_1, x_2, x_3, x_4]^{20}[x, x_1, x_2, x_3, x_4, [x, x_1, x_2, x_3]]^{190}\\
=&1.
\end{aligned}$$

If every nilpotent 3-Engel group were to satisfy the identity

$$[x^{10}, x_1, \cdots, x_d] = 1$$

for some $d \in \mathbb{N}$, then for any finite 3-Engel 2-group G we would have that $G/Z_d(G)$ is of exponent 2 and thus abelian. Hence we would get that $G' \leq Z_d(G)$ and the nilpotency class of finite 3-Engel 2-groups would be bounded above by $d + 1$. We now show that this is not the case. In fact we will show that for all $n \in \mathbb{N}$ and primes $p < n$ there are finite n-Engel p-groups of arbitrarily large class.

Let $n, t \in \mathbb{N}$ and $p < n$ be a prime. Consider the group

$$G = C_p \operatorname{wr} C_p^t = \prod_{g \in C_p^t} C_p^g \rtimes C_p^t.$$

Let $B = \prod_{g \in C_p^t} C_p^g$ be the base group and $M = C_p^t = \langle a \rangle^t$ be the group acting on it.

For each $g, h \in B$ and $x \in M$, we have, as B is abelian,

$$[g, \underbrace{hx, \ldots, hx}_{p \text{ times}}] = [g, \underbrace{x, \ldots, x}_{p \text{ times}}] = g^{(-1+x)^p} = g^{(-1)^p + x^p} = g^{(-1)^p + 1} = 1.$$

Also, for $y \in M$,

$$[gy, hx] = [g, hx]^y [y, hx] = [g, hx]^y [y, x][y, h]^x = [g, hx]^y [y, h]^x \in B.$$

Thus G is a $(p+1)$-Engel group and in particular an n-Engel p-group. However, if we let $e_i \in M$ be the element with a in coordinate i, but 1 in all the other coordinates and we consider $a^1 \in B$, then $[a^1, e_1, e_2, \ldots, e_t] = a^{(-1+e_1)\cdots(-1+e_t)} \neq 1$, as, for example, the C_p^1 part of this is $a^{(-1)^t} \neq 1$. Taking $p = 2$ and $n = 3$ gives $f_3(3) = e_3(3) = 20$.

Finally for 3-Engel groups we consider $d_4(3)$ and $e_4(3)$. We will show that $d_4(3) = c_4(3) = 5$ and $e_4(3) = f_4(3) = 20$. Suppose for a contradiction that this is not the case. We have already seen that 2 and 5 divide $e_4(3)$. Thus every 3-Engel 2-group would satisfy $\gamma_m(G)^2 = \{1\}$, for some $m > 5$. Let G be an arbitrary 3-Engel 2-group. Then $\gamma_m(G)$ would be abelian. Gupta and Newman [17] have shown that 5-torsion-free 3-Engel groups satisfy the identity

$$[a, b, c, [d, e], f] = 1.$$

It follows that, modulo higher multiweights,

$$1 = [d, e, [a, b, c], f] = [d, e, [a, b], c, f].$$

Since $\gamma_m(G)$ is abelian we have, modulo higher multiweights,

$$1 = [a_1, \ldots, a_m, [b_1, \ldots, b_m]] = [a_1, \ldots, a_m, b_m, b_{m-1}, \ldots, b_3, [b_1, b_2]].$$

Since we also have the identity $1 = [a, b, c, d, e, f]^2$ as mentioned earlier, it follows that for commutators in G of weight at least $2m$ we can swap any two entries that aren't either of the first two, modulo higher multiweights. To get a contradiction we show that if $F = \langle x_1, \ldots, x_{2m-2} \rangle \leq G$, then F is nilpotent of class at most $2m - 1$. This contradicts an example of C. K. Gupta [16] of a d-generator 3-Engel group of exponent 4 and nilpotency class at least $d + 2$, for arbitrary $d > 1$.

It remains to see that a commutator $g = [g_1, \ldots, g_{2m}]$ is trivial, where $g_1, \ldots, g_{2m} \in$

$\{x_1, \ldots, x_{2m-2}\}$. As commutators with a triple entry are trivial we may assume that there are double entries of x_1 and x_2 and at least a single entry of x_3 in g. If x_3 is not the first entry, then by $[g_1, \ldots, g_i, x_3] = [x_3, [g_1, \ldots, g_i]]^{-1}$ we can write g as a product of commutators with first entry x_3. By moving the entries, apart from the first two, we can assume that g is of the form $[x_3, x_1, x_1, x_2, x_2, \ldots]$ or $[x_3, g_j, x_1, x_1, x_2, x_2, \ldots]$. Expanding $[a, x_1, x_1x_2, x_1x_2, x_1x_2] = 1$ gives

$$1 = [a, x_1, x_1, x_2, x_2][a, x_1, x_2, x_1, x_2][a, x_1, x_2, x_2, x_1].$$

Replacing a with x_3 or $[x_3, g_j]$ and moving entries of x_2 gives $g^3 = 1$, and as $g^2 = 1$ we have the desired result. This completes the proof that $d_4(3) = 5$ and $e_4(3) = 20$ and finishes the case $n = 3$.

For 4-Engel groups it is known that $c_2(4) = 7$ [14, 36] and that the prime divisors of $e_3(4)$, $f_3(4)$, $e_4(4)$ and $f_4(4)$ are 2, 3 and 5. The exact values are unknown, as are the values of $d_3(4)$, $d_4(4)$ and $c_1(d, 4)$.

3.5 n-Engel p-groups

We now state some results on the structure of n-Engel p-groups which were proved by Abdollahi and Traustason [1]. These will be generalised in the Chapter 4. Let p be a prime. The first theorem says that the nilpotency class of finite n-Engel powerful p-groups is bounded.

Theorem 3.9. *([1]) There exists a positive integer $s = s(n)$ such that any finite n-Engel powerful p-group is nilpotent of class at most s.*

For each finite n-Engel p-group we can find a useful subgroup that is powerful. Let $r = r(n, p)$ be the integer satisfying $p^{r-1} < n \leq p^r$.

Theorem 3.10. *([1]) Let G be a finite n-Engel p-group.*
(a) If p is odd, then G^{p^r} is powerful.
(b) If $p = 2$, then $(G^{2^r})^2$ is powerful.

As a corollary of Theorems 3.9 and 3.10 we then have

Theorem 3.11. *([1]) Let G be a locally finite n-Engel p-group.*
(a) If p is odd, then G^{p^r} is nilpotent of n-bounded class.
(b) If $p = 2$, then $(G^{2^r})^2$ is nilpotent of n-bounded class.

Since locally finite p-groups are nilpotent, for such an n-Engel group G we could apply Theorems 3.3 and 3.4 to get integers $c_3 = c_3(n)$, $c_4 = c_4(n)$, $r_3 = r_3(n)$ and $r_4 = r_4(n)$ such that $G^{p^{r_3}} \leq Z_{c_3}(G)$ and $\gamma_{c_4}(G)^{p^{r_4}} = \{1\}$. However, the r given in Theorem 3.11 gives a specific bound and is close to being the best bound. Let t be the smallest positive integer such that G^{p^t} is nilpotent of n-bounded class, for every locally finite n-Engel p-group G. Then $t \in \{r-1, r\}$ if p is odd and $t \in \{r-1, r, r+1\}$ if $p = 2$. This is clearly true if $r = 1$. To see this for $r > 1$ consider the locally finite p-group

$$G = C_p \operatorname{wr} C^{\infty}_{p^{r-1}} = \prod_{g \in C^{\infty}_{p^{r-1}}} C_p^g \rtimes C^{\infty}_{p^{r-1}}.$$

Let $B = \prod_{g \in C^{\infty}_{p^{r-1}}} C_p^g = \prod_{g \in C^{\infty}_{p^{r-1}}} \langle b \rangle^g$ be the base group and $M = C^{\infty}_{p^{r-1}} = \langle a \rangle^{\infty}$ be the group acting on it. Here B is an infinite direct product and we allow infinitely many non-trivial coordinates. As M is abelian and B is normal in G, G' is in B. As B is abelian we have, for $g, h \in B$ and $x \in M$,

$$[g, \underbrace{hx, \ldots, hx}_{p^{r-1}\text{ times}}] = [g, \underbrace{x, \ldots, x}_{p^{r-1}\text{ times}}] = g^{(-1+x)^{p^{r-1}}} = g^{(-1)^{p^{r-1}} + x^{p^{r-1}}} = g^{(-1)^p + 1} = 1.$$

Hence G is $(p^{r-1}+1)$-Engel and in particular is n-Engel. Let $m = (a^p, 1, 1, \ldots) \in M$. Then

$$(b^1 m)^{p^{r-2}} = b^1 b^{m^{-1}} b^{m^{-2}} \cdots b^{m^{-p^{r-2}+1}} m^{p^{r-2}} = b^1 b^{m^{-1}} b^{m^{-2}} \cdots b^{m^{-p^{r-2}+1}}.$$

Let this element be f and let $e_i \in M$ be the element with $a^{p^{r-2}}$ in coordinate i and 1 in all other coordinates. Then we have, for any $l \geq 2$, $[f, e_2, e_3, \ldots, e_l] \neq 1$, as the C_p^1 part of this is $b^{(-1)^{l-1}} \neq 1$, for example. Note that $f \in G^{p^{r-2}}$ and thus $G^{p^{r-2}}$ is not nilpotent, which along with Theorem 3.11 gives the possible values of t.

3.6 n-Engel p-groups for small n

In this section we consider Theorems 3.9 and 3.11 for small values of n. Let $s(n,p)$ be the smallest positive integer such that every finite n-Engel powerful p-group is nilpotent of class at most $s(n,p)$. Also, let $c(n,p)$ and $f(n,p)$ be positive integers such that $G^{p^{f(n,p)}}$ is nilpotent of class at most $c(n,p)$, for all locally finite n-Engel p-groups G. We wish to find the value of $s(n,p)$ and the best possible values for $c(n,p)$ and $f(n,p)$, for small n.

The case $n = 1$ is just the abelian groups and we have $s(1,p) = 1$, for any prime

p. We can also take $c(1,p)=1$ and $f(1,p)=0$ and these are clearly best possible. We now turn to the case $n=2$. As stated in Section 3.4 we have the identities

$$[x^3, x_1, x_2] = 1,$$
$$[x, x_1, x_2, x_3] = 1.$$

Hence, when $p \neq 3$ we have $s(2,p) \leq 2$ and we have $s(2,3) \leq 3$. We also have from these that the best possible value of $f(2,p)$ is 0 for all p.

We now see that $s(2,p) \geq 2$ for any prime p. First for $p \neq 2$ consider the group $C_{p^2} \rtimes C_p$, where if $C_{p^2} = \langle a \rangle$ and $C_p = \langle b \rangle$, then C_p acts on C_{p^2} via $a^b = a^{1+p}$. Then $[a,b] = a^p$ and $[a,b,b] = [a^p, b] = a^{p^2} = 1$. Thus this is a 2-Engel powerful p-group, but it is not abelian. For $p = 2$ consider instead the group $C_8 \rtimes C_2$, where if $C_8 = \langle a \rangle$ and $C_2 = \langle b \rangle$, then C_2 acts on C_8 via $a^b = a^5$. Then $[a,b] = a^4$ and $[a,b,b] = [a^4, b] = a^{16} = 1$. Hence the result holds for $p = 2$. We thus have $s(2,p) = 2$ for $p \neq 3$. This also gives that for $p \neq 3$, when $f(2,p) = 0$ the best possible value of $c(2,p)$ is 2.

We would now like to find the best possible value for $c(2,p)$. We will see that it is 2 for every p. From the identities given above we have, for any p, that we may take $c(2,p) = 2$. Thus it remains to give an example to show that $c(2,p)$ cannot be taken to be 1 for any value of $f(2,p)$. For this let $t \in \mathbb{N}$ and consider the subgroup U of $GL(3, \mathbb{Z}_{p^t})$ consisting of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c \in \mathbb{Z}_{p^t}$. Calculating the commutator of three such matrices shows that U is nilpotent of class at most 2. In particular U is a finite 2-Engel p-group. Consider the matrices

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

in U. Then

$$[X, Y] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence, for any $0 \leq i < t/2$,

$$[X^{p^i}, Y^{p^i}] = [X,Y]^{p^{2i}} = \begin{pmatrix} 1 & 0 & p^{2i} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq 1.$$

Since t was arbitrary, this shows that the best possible value of $c(2,p)$ is 2, for every p.

When $c(2,p) = 2$ we can take $f(2,p) = 0$ for $p \neq 3$. Hence these values are both best possible. We also see from the identities that we can take $(c(2,3), f(2,3)) = (2,1)$ or $(3,0)$. In [29] and [37], Moravec and Traustason classify the powerful 2-Engel groups of class 3. These are 3-groups and hence show that we cannot take $(c(2,3), f(2,3)) = (2,0)$. Thus the pairs $(c(2,3), f(2,3))$ given above are the best possible. This also shows that $s(2,3) = 3$. This finishes the case $n = 2$.

We now consider the case $n = 3$ and give upper bounds for each value. We find the best possible values for $f(3,p)$. We have seen in Section 3.4 that the identity

$$[x^{20}, x_1, x_2, x_3, x_4] = 1 \tag{3.1}$$

holds in any 3-Engel group. First consider the case $p \neq 2, 5$. It follows that we can take $c(3,p) = 4$ and that the best possible value for $f(3,p)$ is 0. It also follows that $s(3,p) \leq 4$, for $p \neq 2, 5$.

Next we consider the case $p = 2$. From the identity (3.1) we see that $s(3,2) \leq 5$. In this case $r = 2$ and hence by the discussion in Section 3.5 we have that $f(3,2) \geq 1$. We now show that we can have $f(3,2) = 1$ and hence this will be best possible. Let G be a 3-Engel 2-group and let $g, g_1, g_2, g_3, g_4 \in G$. In [17], Gupta and Newman show that every n-generator 5-torsion-free 3-Engel group is nilpotent of class at most $n+2$. Hence $\langle g, g_1, g_2, g_3, g_4 \rangle$ is nilpotent of class at most 7. It is also shown that the following two identities hold in these groups.

$$[x, x_1, x_2, x_3, x_4]^4 = 1,$$
$$[x, x_1, x_2, x_3, x_4, x_5]^2 = 1.$$

Using these we have

$$\begin{aligned}
[g^2, g_1^2, g_2^2, g_3^2, g_4^2] =&[g^2, g_1^2, g_2^2, g_3^2, g_4]^2[g^2, g_1^2, g_2^2, g_3^2, g_4, g_4]\\
=&[[g^2, g_1^2, g_2^2, g_3]^2, g_4]^2[g^2, g_1^2, g_2^2, g_3, g_3, g_4]^2[g^2, g_1^2, g_2^2, g_3^2, g_4, g_4]\\
=&[g^2, g_1^2, g_2^2, g_3, g_4]^4[[g^2, g_1^2, g_2^2, g_3]^2, g_4, g_4][g^2, g_1^2, g_2^2, g_3, g_3, g_4, g_4]\\
=&[g^2, g_1^2, g_2^2, g_3, g_4, g_4]^2[[g^2, g_1^2, g_2]^2, g_3, g_3, g_4, g_4]\\
=&[g^2, g_1^2, g_2, g_3, g_3, g_4, g_4]^2\\
=&1.
\end{aligned}$$

Thus we have that G^2 is nilpotent of class at most 4 and in particular we can have $f(3, 2) = 1$.

Finally we consider the case $p = 5$. From the identity (3.1) we see that $s(3, 5) \leq 5$ and that we can take $f(3, 5) = 1$ and $c(3, 5) = 4$. The example of Bachmuth and Mochizuki [3] shows that $f(3, 5) = 1$ is best possible.

Chapter 4

Right n-Engel Subgroups

4.1 Introduction

For each theorem on n-Engel groups in Chapter 3, we state and prove the corresponding theorem for normal right n-Engel subgroups. Since an n-Engel group is a right n-Engel subgroup of itself, these theorems will imply the theorems in Chapter 3. Theorems 4.1 - 4.3 are proved in [10] and Theorem 4.4 is proved in [11]. The statements were slightly different in [10] as we will mention in more detail, and the proofs have been adjusted slightly to account for this. Theorems 4.12 - 4.14 are the analogous versions of Theorems 3.9 - 3.11 and are proved in [11], with the proofs given again here.

4.2 Right n-Engel subgroup theorems

Here we state and discuss the analogues of Theorems 3.1 - 3.4. Note that the analogue of the group being nilpotent is that the subgroup is upper central. First we have the analogue of Theorem 3.1.

Theorem 4.1. *Let H be a normal upper central right n-Engel subgroup of a d-generator group G. Then the upper central degree of H is bounded by a function in d and n.*

Here the whole group was assumed to be finitely generated. For the analogue of Theorem 3.2 we don't need to require that the whole group is torsion-free, but only that the right n-Engel subgroup is.

Theorem 4.2. *Let H be a normal upper central right n-Engel subgroup of a group G and suppose that H is torsion-free. Then the upper central degree of H is bounded by a function in n.*

We also have the analogues of Theorems 3.3 and 3.4.

Theorem 4.3. *Let H be a normal upper central right n-Engel subgroup of a group G. Then there exist positive integers $c = c(n)$, $e = e(n)$, only depending on n, such that $H^e \leq Z_c(G)$.*

Theorem 4.4. *Let H be a normal upper central right n-Engel subgroup of a group G. Then there exist positive integers $c = c(n)$, $e = e(n)$, only depending on n, such that $[H,_c G]^e = \{1\}$.*

As in the n-Engel case we have that Theorems 4.3 and 4.4 each imply Theorem 4.2. Theorem 4.4 clearly implies Theorem 4.2. To see that Theorem 4.3 implies Theorem 4.2 we again use Lemma 2.15. Let $c = c(n)$ and $e = e(n)$ be such that for every group G and every normal upper central right n-Engel subgroup H of G, $H^e \leq Z_c(G)$. Suppose that H is a normal upper central right n-Engel subgroup of a group G and H is torsion-free. By Lemma 2.15 we have that the image of H under the natural homomorphism from G to $G/Z_c(G)$ is torsion-free. Thus by Theorem 4.3 we have $H \leq Z_c(G)$. So Theorem 4.3 does imply Theorem 4.2. As in the n-Engel group case we will use Theorem 4.2 to prove Theorem 4.3. The proof of Theorem 4.4 will also make use of Theorem 4.2.

Theorems 4.1 - 4.4 give rise to integer valued functions in a similar way to Theorems 3.1 - 3.4. Thus we have the corresponding definition to Definition 3.5.

Definition 4.5. Let d and n be positive integers.
(i) Define $\hat{c}_1(d,n)$ to be the smallest positive integer such that for every d-generator group G, every normal upper central right n-Engel subgroup of G has upper central degree at most $\hat{c}_1(d,n)$.
(ii) Define $\hat{c}_2(n)$ to be the smallest positive integer such that for every group G, every normal upper central right n-Engel subgroup of G that is torsion-free has upper central degree at most $\hat{c}_2(n)$.
(iii) Define $\hat{c}_3(n)$ to be the smallest positive integer such that for every group G, every normal upper central right n-Engel subgroup H of G satisfies $H^{e(n)} \leq Z_{\hat{c}_3(n)}(G)$ for some positive integer $e(n)$. Then define $\hat{f}_3(n)$ to be the smallest possible value of $e(n)$ here. Define $\hat{e}_3(n)$ to be the smallest positive integer such that for every group G, every normal upper central right n-Engel subgroup H of G satisfies $H^{\hat{e}_3(n)} \leq Z_{c(n)}(G)$ for some positive integer $c(n)$. Then define $\hat{d}_3(n)$ to be the smallest possible value of $c(n)$ here.
(iv) Define $\hat{c}_4(n)$ to be the smallest positive integer such that for every group G, every normal upper central right n-Engel subgroup H of G satisfies $[H,_{\hat{c}_4(n)} G]^{e(n)} = \{1\}$ for some positive integer $e(n)$. Then define $\hat{f}_4(n)$ to be the smallest possible value of $e(n)$ here. Define $\hat{e}_4(n)$ to be the smallest positive integer such that every nilpotent n-Engel

group G satisfies $[H,_{c(n)} G]^{\hat{e}_4(n)} = \{1\}$ for some positive integer $c(n)$. Then define $\hat{d}_4(n)$ to be the smallest possible value of $c(n)$ here.

In [10] Theorems 4.1 - 4.3 were stated slightly differently using the term residually hypercentral, which is defined as follows.

Definition 4.6. Let H be a subgroup of a group G. We call H *residually hypercentral (in G)* if

$$\cap_{i=0}^{\infty}[H,_i G] = \{1\}.$$

Recall from Remark 3.6 that Theorems 3.1 - 3.4 still held when the group was assumed to be residually nilpotent instead of nilpotent. The situation here is similar, in that all four theorems above are equivalent to the theorems with the condition H is upper central replaced by the condition H is residually hypercentral. Clearly if H is upper central, then it is residually hypercentral. Thus the theorems with H residually hypercentral imply those above. For the converse suppose that H is a residually hypercentral normal right n-Engel subgroup of a group G. For arbitrary $i \in \mathbb{N}$, quotient G out by $[H,_i G]$. In this quotient H is an upper central normal right n-Engel subgroup. Theorems 4.3 and 4.4 apply and as i was arbitrary we get $[H^{\hat{f}_3(n)},_{\hat{c}_3(n)} G]$ and $[H,_{\hat{c}_4(n)} G]^{\hat{f}_4(n)}$ are in $\cap_{i=0}^{\infty}[H,_i G]$ and hence are trivial. Note that if G is a d-generator group so is $G/[H,_i G]$. Hence Theorem 4.1 applies in this case and we get $[H,_{\hat{c}_1(n)} G] \leq \cap_{i=0}^{\infty}[H,_i G] = \{1\}$. For Theorem 4.2 suppose that H is torsion-free and residually hypercentral. We have seen that Theorem 4.4 holds when H is residually hypercentral and not just upper central. Thus, as H is torsion-free, we have $[H,_{\hat{c}_4(n)} G] = \{1\}$.

Recall that right n-Engel elements of finite groups and finitely generated solvable groups are contained in the hypercentre [4, 5]. In a finite group the hypercentre is a term of the upper central series. If G is a finitely generated solvable group, and H is a normal right n-Engel subgroup of G, then $H \leq \cup_{i=1}^{\infty} Z_i(G)$. For $h \in H$, we have that $\langle h \rangle^G$ is then upper central. Hence, by Theorem 4.1, $\langle h \rangle^G$ is upper central with upper central degree at most $\hat{c}_1(d,n)$, where d is the rank of G. Hence H is also upper central with upper central degree at most $\hat{c}_1(d,n)$. Thus normal right n-Engel subgroups of finitely generated solvable groups are upper central.

Theorem 4.1 is a generalisation of Theorem 3.1 and as such it gives a positive solution to an analogous problem to the restricted local nilpotence problem, stated in Section 1.1, for right n-Engel subgroups. A natural question to ask would be whether the analogous statement of the local nilpotence problem holds. That is, for each positive integer n, is every right n-Engel subgroup of a finitely generated group upper central? In fact it doesn't hold, as the following counterexample shows.

In [2] Adjan proves that the Burnside groups $B(r, n)$, that is the free groups of exponent n on r generators, are infinite for $r > 1$ and odd $n \geq 665$. Let p be a prime such that $p > 665$. Let

$$G = C_p \operatorname{wr} B(2, p) = \bigoplus_{g \in B(2,p)} C_p^g \rtimes B(2, p).$$

Let $M = \bigoplus_{g \in B(2,p)} C_p^g$ be the base group. Note that M is an infinite direct sum and thus we only allow elements that have finitely many non-trivial coordinates. If $C_p^1 = \langle a \rangle$, then $G = \langle a, B(2, p) \rangle$ is finitely generated. For each $g, h \in M$ and $x \in B(2, p)$, we have

$$[g, \underbrace{hx, \ldots, hx}_{p \text{ times}}] = [g, \underbrace{x, \ldots, x}_{p \text{ times}}] = g^{(-1+x)^p} = g^{(-1)^p + x^p} = g^{-1+1} = 1.$$

Thus M is a right p-Engel subgroup of G. Since $B(2, p)$ is infinite we can choose the following sequence of elements, which we define recursively. First let $1 \neq g_1 \in B(2, p)$. Let $X_i = \{g_{j_1} \cdots g_{j_m} : m \geq 0 \text{ and } 1 \leq j_1 < j_2 < \ldots < j_m \leq i\}$ and $X_i^{-1} = \{x^{-1} : x \in X_i\}$. Then we choose $g_{i+1} \in B(2, p)$ such that $g_{i+1} \notin X_i \cup X_i^{-1}$. For $t \in \mathbb{N}$ consider

$$[a, g_1, \ldots, g_t] = a^{(-1+g_1)\cdots(-1+g_t)}.$$

If this were trivial, then g_t would equal some distinct product $g_{j_1} \cdots g_{j_m}$, where $m \geq 0$ and $1 \leq j_1 < j_2 < \ldots < j_m \leq t$. By construction this could only happen if $m \geq 2$ and $j_m = t$. But then we would have $g_{j_1} \cdots g_{j_{m-1}} = 1$. Since $g_{j_{m-1}} \neq (g_{j_1} \cdots g_{j_{m-2}})^{-1}$ this cannot be the case. Hence M is not upper central in G.

We now turn to the proofs of Theorems 4.1 - 4.4. Our main tools for the proofs of Theorems 4.1 and 4.2 are the two results of Zel'manov on Lie rings, Theorems 2.11 and 2.12. These are also the main tools in the proofs of Theorems 3.1 and 3.2. For the proofs of Theorems 4.3 and 4.4 we follow a similar method to the proofs of Theorems 3.3 and 3.4.

4.3 Proof of Theorem 4.1

Let $G = \langle f_1, f_2, \ldots, f_d \rangle$ be any d-generator group and let H be a normal right n-Engel subgroup that is upper central. We will apply Theorem 2.11 and for this we will associate a Lie ring L to the pair (H, G). First we define the chains $(H_i)_{i=0}^{\infty}$ and $(G_i)_{i=1}^{\infty}$, where the latter is the lower central series of G, that is $G_i = \gamma_i(G)$, and $H_i = [H,_i G]$, defined recursively by $[H,_0 G] = H$ and $[H,_{i+1} G] = [[H,_i G], G]$.

Let $A_i = H_i/H_{i+1}$ and $L_i = G_i/G_{i+1}$. From these we form the abelian groups

$$A = A_0 \oplus A_1 \oplus \cdots, \qquad L_G = L_1 \oplus L_2 \oplus \cdots.$$

We consider A as an abelian Lie ring, that is A has zero multiplication, and we let L_G be the usual associated Lie ring of G. Thus if $x = aG_{i+1} \in L_i$ and $y = bG_{j+1} \in L_j$, then their Lie product is $xy = [a, b]G_{i+j+1} \in L_{i+j}$. This product is extended linearly onto L_G. We define multiplication between elements of A and L_G as follows. If $u = hH_{i+1} \in A_i$ and $x = aG_{j+1} \in L_j$, then we let $ux = [h, a]H_{i+j+1}$. We need to check that this is well-defined and that the product is in A.

Lemma 4.7. *If $hH_{i+1} \in A_i$ and $aG_{j+1} \in L_j$, then $[h, a]H_{i+j+1} \in A_{i+j}$. Further, the above multiplication is well-defined.*

Proof. For the first part, since $h \in H_i$ and $a \in G_j$, it suffices to see that $[H_i, G_j] \le H_{i+j}$. We prove this by induction on $j \ge 1$. The case $j = 1$ is trivial, by the definition of $(H_i)_{i=0}^{\infty}$. Now suppose that this is true for $j = k$. Then,

$$\begin{aligned}
[H_i, G_{k+1}] &= [G_{k+1}, H_i] \\
&= [G_k, G, H_i] \\
&\le [G_k, H_i, G][G_k, [G, H_i]], \text{ by the 3 subgroups lemma (Lemma 2.4),} \\
&\le [H_{i+k}, G][G_k, H_{i+1}] \\
&\le H_{i+k+1}.
\end{aligned}$$

To see that the multiplication is well-defined, let $a_1, a_2 \in H_i$ be such that $a_1 \in a_2H_{i+1}$ and $b_1, b_2 \in G_j$ be such that $b_1 \in b_2G_{j+1}$. Then, modulo H_{i+j+1},

$$\begin{aligned}
[a_1, b_1] &= [a_2h, b_2g], \text{ for some } h \in H_{i+1},\ g \in G_{j+1}, \\
&= [a_2h, g][a_2h, b_2][a_2h, b_2, g] \\
&= [a_2, g][a_2, b_2][a_2, b_2, h][h, b_2][a_2, b_2, g] \\
&= [a_2, b_2], \text{ by the first part of this lemma.}
\end{aligned}$$

□

We now extend the multiplication linearly to get an action from L_G on A. As A is abelian, we have for all $a, b \in A$ and $x \in L$ that $abx = 0 = (ax)b + a(bx)$ and thus the right multiplication by x is a derivation. Also, if $u = hH_{i+1} \in A_i$, $x = aG_{j+1} \in L_j$ and $y = bG_{k+1} \in L_k$, then

$$u(xy) = [h, [a, b]]H_{i+j+k+1} = [h, a, b][h, b, a]^{-1}H_{i+j+k+1} = uxy - uyx.$$

Hence right multiplication on A by L_G is a Lie ring homomorphism from L_G to Der$A$. Thus the action from L_G on A gives rise to a semidirect product $M = A \rtimes L_G$ and also to $L = A \rtimes L_G/C_{L_G}(A)$, where $C_{L_G}(A)$ is the centraliser of A in L_G. The latter is well-defined, since if $x \in y + C_{L_G}(A)$, then $ax = ay\ \forall a \in A$.

The aim is to show that $AL_G^m = \{0\}$, for some (d, n)-bounded integer m. This would imply that $A_m = A_0L_1^m = \{0\}$ and thus that $H_m = H_{m+1}$. As H is upper central it would then follow that $H_m = [H,_m G] = \{1\}$. So it remains to show that in M we have $AL_G^m = \{0\}$, or equivalently that in L we have $A(L_G/C_{L_G}(A))^m = \{0\}$, for some (d, n)-bounded integer m. We want to apply Theorem 2.11 and for this we need to introduce some notation.

For any positive integer m we let $C_m = \{1, 2, \ldots, m\}$, $\mathcal{P}(C_m)$ be the powerset of C_m, and

$$\mathcal{R}_m = \{(S, T) \in \mathcal{P}(C_m) \times \mathcal{P}(C_m) :\ S \cup T = \{1, \ldots, m\} \text{ and } S \cap T = \emptyset\}.$$

We will use the following formula, which holds in any Lie ring and we prove by induction on $m \geq 1$.

Lemma 4.8. *Let $b, y, u_1, \ldots, u_m$ be elements of a Lie ring. Then,*

$$b(yu_1 \cdots u_m) = \sum_{(S,T)\in\mathcal{R}_m} (-1)^{|T|} b\bar{u}_T y u_S,$$

where, if $S = \{i_1, \ldots, i_r\}$, $i_1 < i_2 < \ldots < i_r$, and $T = \{j_1, \ldots, j_k\}$, $j_1 < j_2 < \ldots < j_k$, then $b\bar{u}_T y u_S = bu_{j_k}u_{j_{k-1}} \cdots u_{j_1} y u_{i_1} \cdots u_{i_r}$.

Proof. The case $m = 1$ follows directly from the Jacobi identity, since

$$b(yu_1) = -y(u_1b) - u_1(by) = -bu_1y + byu_1.$$

Now suppose that the formula holds for $m = l$. Then, using the case $m = 1$,

$$\begin{aligned}
b(yu_1 \cdots u_{l+1}) =& b(yu_1 \cdots u_l)u_{l+1} - bu_{l+1}(yu_1 \cdots u_l) \\
=& \sum_{(S,T)\in\mathcal{R}_l} (-1)^{|T|} b\bar{u}_T y u_S u_{l+1} + \sum_{(S,T)\in\mathcal{R}_l} (-1)^{|T|+1} bu_{l+1}\bar{u}_T y u_S \\
=& \sum_{(S,T)\in\mathcal{R}_{l+1}} (-1)^{|T|} b\bar{u}_T y u_S,
\end{aligned}$$

which completes the induction. □

For $\sigma \in S_m$ we use $(b\bar{u}_T y u_S)^\sigma$ for

$$bu_{\sigma(j_k)}u_{\sigma(j_{k-1})} \cdots u_{\sigma(j_1)} y u_{\sigma(i_1)} \cdots u_{\sigma(i_r)}.$$

We will furthermore use the following useful notation. If $u, v_1, \ldots, v_m$ are elements in a Lie ring, then

$$u\{v_1, \ldots, v_m\} = \sum_{\sigma\in S_m} uv_{\sigma(1)} \cdots v_{\sigma(m)}.$$

We now show that L satisfies the first condition of Theorem 2.11. Notice that we don't use that G is finitely generated in this lemma. We will therefore be able to use the lemma for the proof of Theorem 4.2 as well.

Lemma 4.9. *Let $a = hH_{i+1} \in A_i, x = gG_{j+1} \in L_j$ and $x_i = g_iG_{\beta(i)+1} \in L_{\beta(i)}$ for $i = 1, \ldots, 3n-2$. Then*
(a) $a\{x_1, \ldots, x_n\} = 0$;
(b) $x\{x_1, \ldots, x_{2n-1}\} \in C_{L_G}(A)$;
(c) $x\{a, x_1, \ldots, x_{3n-2}\} = 0$.

Proof. As h is a right n-Engel element we have $[h,_n y] = 1$ for all $y \in G$. For arbitrary elements $y_1, y_2, \ldots, y_n \in G$ we have that $[h,_n y_1y_2 \cdots y_n] = 1$. Expanding this commutator and using Hall's collection process we see that

$$\prod_{\sigma\in S_n} [h, y_{\sigma(1)}, \ldots, y_{\sigma(n)}] = u,$$

where u is a product of commutators with multiweight strictly higher than $(1, \ldots, 1)$ in $h, y_1, \ldots, y_n$. In particular it follows that

$$\prod_{\sigma\in S_n} [h, g_{\sigma(1)}, \ldots, g_{\sigma(n)}] \in H_{i+\beta(1)+\cdots+\beta(n)+1}.$$

Hence we get in M that

$$\sum_{\sigma \in S_n} a x_{\sigma(1)} \cdots x_{\sigma(n)} = 0,$$

which proves (a).

We now move to (b). It suffices to show that $a(x\{x_1 \cdots x_{2n-1}\}) = 0$. Using Lemma 4.8 this is equal to

$$\sum_{(S,T) \in \mathcal{R}_{2n-1}} (-1)^{|T|} \sum_{\sigma \in S_{2n-1}} (a \bar{x}_T x x_S)^{\sigma}.$$

Since for each pair (S,T) we either have $|S| \geq n$ or $|T| \geq n$ it now follows from (a) that this sum is equal to 0. This proves (b).

For (c) notice that

$$x\{a, x_1, \ldots, x_{3n-2}\} = \sum_{r=0}^{3n-2} \sum_{\sigma \in S_{3n-2}} x x_{\sigma(1)} \cdots x_{\sigma(r)} a x_{\sigma(r+1)} \cdots x_{\sigma(3n-2)}$$

and as either $r \geq 2n-1$ or $3n-2-r \geq n$ it follows from (a) and (b) that the sum is trivial. $\square$

As A is abelian every product in L with two occurrences of an element from A is trivial. It follows from Lemma 4.9 and multilinearity that L satisfies the identity

$$\sum_{\sigma \in S_{3n-1}} u v_{\sigma(1)} \cdots v_{\sigma(3n-1)} = 0. \tag{4.1}$$

Thus L satisfies the first condition of Theorem 2.11. We now turn to the second condition.

We take $h \in H$ and the generators $f_1, \ldots, f_d$ of G. Consider the elements $b = h[H,G] \in A_0$ and $y_1 = f_1[G,G], \ldots, y_d = f_d[G,G] \in L_1$. Notice that $y_1, \ldots, y_d$ generate L_G. We consider the subalgebra N of L generated by b and $y_1 + C_{L_G}(A), \ldots, y_d + C_{L_G}(A)$. Now, N satisfies the linearised Engel identity (4.1) and in order to apply Theorem 2.11 we will show that N also satisfies the second condition. In particular we show that, for any Lie product v in the generators $b, y_1 + C_{L_G}(A), \ldots, y_d + C_{L_G}(A)$, we have that

$$u v^{2n-1} = 0 \tag{4.2}$$

for all $u \in N$. Then it will follow from Theorem 2.11 that N is nilpotent of (d,n)-

bounded class m. In particular we can conclude that

$$bx_1x_2\cdots x_m = 0$$

for all $x_1, \ldots, x_m \in L_G/C_{L_G}(A)$ and, as $b \in A_0$ is arbitrary, it follows that $A_m = A_0(L_G/C_{L_G}(A))^m = \{0\}$. Hence $[H,_m G] = [H,_{m+1} G]$ and as H is upper central, it then follows that $[H,_m G] = \{1\}$. To finish the proof of Theorem 4.1, it thus suffices to show that equation (4.2) holds in N.

In fact we prove something stronger, namely that (4.2) holds for all $u \in L$. As A is an abelian ideal of L it is clear that this holds if v has more than 1 occurrence of b. Now suppose that v has exactly one occurrence of b. Then $uv^n = 0$ when $n \geq 2$ again for the reason that A is abelian. So in this case we can assume that $n = 1$. But then $H \leq Z(G)$ and thus $AL = \{0\}$, which implies that $uv \in LA = \{0\}$. So we may assume that v is a Lie product in the generators $y_1 + C_{L_G}(A), \ldots, y_d + C_{L_G}(A)$. Then $v = w + C_{L_G}(A)$, where $w \in L_j$ for some $j \geq 1$, say $w = gG_{j+1}$. We first prove that

$$av^n = 0 \tag{4.3}$$

for all $a \in A$. By linearity we can suppose that $a \in A_i$ for some $i \geq 0$, say $a = kH_{i+1}$. Since $k \in H_i$ is a right n-Engel element we have

$$av^n = aw^n = [k,_n g]H_{i+nj+1} = 1H_{i+nj+1} = 0.$$

It now only remains to show that $xv^{2n-1} = 0$ for all $x \in L_G/C_{L_G}(A)$ or equivalently that $xw^{2n-1} \in C_{L_G(A)}$ for all $x \in L_G$. But, by Lemma 4.8,

$$a(xw^{2n-1}) = \sum_{r=0}^{2n-1} (-1)^r \binom{2n-1}{r} aw^r xw^{2n-1-r}$$

and for each r we have that either $r \geq n$ or $2n-1-r \geq n$. Thus it follows from (4.3) that $a(xw^{2n-1}) = 0$ for all $a \in A$. This finishes the proof of Theorem 4.1.

4.4 Proof of Theorem 4.2

Let G be a group with a torsion-free normal right n-Engel subgroup H that is contained in the lth term of the upper central series of G. We want to apply Theorem 2.12 and in order to do this we will associate to the pair (H, G) a certain torsion-free Lie ring. We modify the construction that we used in the proof of Theorem 4.1 and replace the terms $[H,_i G]$ and $\gamma_j(G)$ by their isolators, as defined in Section 2.5. We consider the two series $(H_i)_{i=0}^{\infty}$ and $(G_i)_{i=1}^{\infty}$, where $H_i = \sqrt[H]{[H,_i G]}$ and $G_i = \sqrt[G]{\gamma_i(G)}$. Since $G/\gamma_i(G)$ and $H/[H,_i G]$ are nilpotent, Lemma 2.18 shows that the terms in these series are subgroups of G. For the Lie ring construction we need the following lemma.

Lemma 4.10. *For all integers $i \geq 0$ and $j \geq 1$, we have $[H_i, G_j] \leq H_{i+j}$.*

Proof. Let $h \in H_i$ and $g \in G_j$ and consider $a = hH_{i+j}$, $b = gH_{i+j}$ in G/H_{i+j}. It suffices to show that $R = \langle a, b\rangle$ is abelian. Let $S = \langle a\rangle^R$. Let r be a positive integer such that $h^r \in [H,_i G]$ and $g^r \in \gamma_j(G)$. Then $[h^r, g^r] \in [[H,_i G], \gamma_j(G)] \leq [H,_{i+j} G]$, by Lemma 4.7, and thus $[a^r, b^r] = 1$. It follows that $[a, b]^{r^2} \in [S,_2 R]$ and thus $[S, R]^{r^2} \leq [S,_2 R]$. Inductively it follows that $[S,_k R]^{r^2} \leq [S,_{k+1} R]$. As $[S,_l R] = 1$ we can conclude that

$$[S, R]^{r^{2(l-1)}} = \{1\}.$$

As $R \leq G/H_{i+j}$ is torsion-free, it follows that $[S, R] = \{1\}$ and thus $[a, b] = 1$. □

As in Section 4.3, we let $A_i = H_i/H_{i+1}$, for $i \geq 0$, and $L_j = G_j/G_{j+1}$, for $j \geq 1$. We then form the abelian groups

$$A = A_0 \oplus A_1 \oplus \cdots, \quad L_G = L_1 \oplus L_2 \oplus \cdots.$$

We consider A as an abelian Lie ring and L_G as the usual associated Lie ring of G with respect to the series $(G_j)_{j=1}^{\infty}$. This is well defined using the fact that $[G_i, G_j] \leq G_{i+j}$, which follows by setting $H = G$ in Lemma 4.10. Defining multiplication between elements of L_G and H as in Section 4.3 and using Lemma 4.10 we again get semidirect products $M = A \rtimes L_G$ and $L = A \rtimes L_G/C_{L_G}(A)$. Notice that in the proof of Lemma 4.9, the fact that G was finitely generated was never used. Lemma 4.9 therefore also holds in the present setting and, as before, we get that L satisfies the linearised $(3n-1)$-Engel identity

$$\sum_{\sigma \in S_{3n-1}} uv_{\sigma(1)}v_{\sigma(2)} \cdots v_{\sigma(3n-1)} = 0.$$

In this setting L is however a torsion-free Lie ring and thus the linearised $(3n-1)$-Engel identity is equivalent to the $(3n-1)$-Engel identity, which is easily seen by setting $v_1 = v_2 = \ldots = v_{3n-1}$. By Theorem 2.12 it then follows that L is nilpotent of n-bounded class m. Hence, for every $k \geq m$, we have in particular that $A_k = A(L_G/C_{L_G}(A))^k = \{0\}$ and thus $\sqrt[H]{[H,_k G]} = \sqrt[H]{[H,_{k+1} G]}$. As $[H,_l G] = \{1\}$ it follows that $\sqrt[H]{[H,_m G]} = \sqrt[H]{[H,_l G]} = \sqrt[H]{\{1\}}$. As H is torsion-free we have that $\sqrt[H]{\{1\}} = \{1\}$. It then follows that $[H,_m G] = \{1\}$. This finishes the proof of Theorem 4.2.

4.5 Proof of Theorem 4.3

In this section we will use Theorem 4.1 and Theorem 4.2 to prove Theorem 4.3. The proof is similar to the proof of the corresponding n-Engel case, Theorem 3.3, which was proved in Section 3.3. We start with a generalised version of Lemma 3.7, from which the proof of Theorem 4.3 will be easily derived.

Lemma 4.11. *Let G be a group with a normal right n-Engel subgroup H that is upper central. Then there exists a positive integer $l = l(n)$ and a non-negative integer $k = k(n)$ such that*

$$[h, g_1, \ldots, g_k]^l = 1$$

for all $h \in H$ and all $g_1, \ldots, g_k \in G$.

Proof. Let $k = k(n) = \hat{c}_2(n)$ be as in Definition 4.5. Consider the relatively free group $F = \langle y, x_1, \ldots, x_k \rangle$, subject to $\langle y \rangle^F$ consisting of right n-Engel elements of F and $\langle y \rangle^F$ being upper central in F of degree $\hat{c}_1(k+1, n)$, with $\hat{c}_1$ as in Definition 4.5. In particular, $\langle y \rangle^F$ is nilpotent and so the torsion elements in $\langle y \rangle^F$ form a normal subgroup, say T. Then $\langle y \rangle^F / T$ is a torsion-free subgroup of F/T. This allows us to use Theorem 4.2 to conclude that $[\langle y \rangle^F, {}_k F] \leq T$ and thus there is some positive integer $l = l(n)$ such that

$$[y, x_1, \ldots, x_k]^l = 1.$$

Let $K = \langle h, g_1, \ldots, g_k \rangle$ and consider a homomorphism ϕ from F to K that maps y to h and x_i to g_i. Then $[h, g_1, \ldots, g_k]^l = \phi([y, x_1, \ldots, x_k]^l) = 1$. This finishes the proof. □

We will now derive Theorem 4.3 from Lemma 4.11. Assume that H and G are as in Lemma 4.11 and let $k = k(n)$ and $l = l(n)$ also be as in Lemma 4.11. Let $h \in H$ and let $g_1, \ldots, g_k \in G$. Then let $E = \langle h, g_1, \ldots, g_k \rangle$ and $K = \langle h \rangle^G$. By Theorem 4.1 we have that $K \leq Z_c(E)$, for some n-bounded number c. We show by reverse induction on $r \in \{0, 1, \ldots, c-k\}$ that $h^{l^{c-k-r}} \in Z_{k+r}(E)$. As $h \in Z_c(E)$ this is clearly true for $r = c - k$. Now suppose that $0 \leq r < c - k$ and that the result holds for larger values of $r \in \{1, \ldots, c-k\}$. By the induction hypothesis we then have that $h^{l^{c-k-r-1}} \in Z_{k+r+1}(E)$. But then we have, for any $e_1, \ldots, e_{k+r} \in E$, that

$$\begin{aligned}[h^{l^{c-k-r}}, e_1, \ldots, e_{k+r}] &= [(h^{l^{c-k-r-1}})^l, e_1, \ldots, e_{k+r}] \\ &= [h^{l^{c-k-r-1}}, e_1, \ldots, e_{k+r}]^l,\end{aligned}$$

which is trivial by Lemma 4.11. This shows that $h^{l^{c-k-r}} \in Z_{k+r}(E)$, which finishes the induction step. Thus the claim holds and in particular for $r = 0$ we get that

$$[h^{l^{c-k}}, g_1, \ldots, g_k] = 1.$$

As $h \in H$ and $g_1, \ldots, g_k \in G$ were arbitrary, this finishes the proof of Theorem 4.3.

4.6 Proof of Theorem 4.4

Let G be a group with normal right n-Engel subgroup H which is upper central. By Lemma 4.11, we know that there exist positive integers $m = m(n)$ and $l = l(n)$ such that, for any $h \in H$ and $g_1, \ldots, g_m \in G$, $[h, g_1, \ldots, g_m]^l = 1$. Fix $h \in H$ and $g_1, \ldots, g_{m+1} \in G$ and let $K = \langle [h, g_1, \ldots, g_m], g_{m+1} \rangle$. Then $K'/[K', K']$ is abelian of exponent dividing l. Let $k = [h, g_1, \ldots, g_m]$. By Theorem 4.1, there exists a positive integer $c = c(n)$ such that $[\langle k \rangle^K,_c K] = \{1\}$. It follows in particular that K' is nilpotent of class at most c and thus K' has exponent dividing l^c. Let e be the smallest positive integer such that $\binom{l^e}{i}$ is divisible by l^c for $i = 1, \ldots, c$ and set $f = l^e$. Let $g = a_1 \cdots a_t$ be any product of commutators of the form $[y, x_1, \ldots, x_m]$, with $y \in H$ and $x_1, \ldots, x_m \in G$. We prove, by induction on t, that $g^f = 1$. If $t = 1$, then this is trivial by Lemma 4.11. Now suppose that $t \geq 2$ and that the induction hypothesis holds for smaller values. Let $z = a_2 \cdots a_t$ and $a = a_1$. Applying the Hall-Petrescu formula (Theorem 2.7), we have that

$$a^f z^f = (az)^f w_2^{\binom{f}{2}} w_3^{\binom{f}{3}} \cdots w_c^{\binom{f}{c}}$$

with $w_i \in \gamma_i(\langle a, z \rangle)$. By inductive hypothesis the left hand side is trivial and by the definition of f every $w_i^{\binom{f}{i}}$ is also trivial, since each w_i is in $\langle a_1, z \rangle'$. Hence $(a_1 \cdots a_t)^f = (az)^f = 1$. This finishes the inductive proof and we conclude that $g^f = 1$ for all $g \in [H,_m G]$. $\Box$

4.7 Right n-Engel subgroups of p-groups

Here we generalise Theorems 3.9 - 3.11 by proving the analogous results for normal right n-Engel subgroups. The analogue to the group being a powerful p-group is that the subgroup is powerfully embedded in the group. First we state the three analogous theorems.

Theorem 4.12. *There exists a positive integer $s = s(n)$, dependent only on n, such that, for any finite p-group G and right n-Engel subgroup H which is powerfully embedded in G, $[H,_{s(n)} G] = 1$.*

As for Theorems 3.10 and 3.11 we let r be the integer satisfying $p^{r-1} < n \leq p^r$.

Theorem 4.13. *Let G be a finite p-group and H be a normal right n-Engel subgroup of G.*
(a) If p is odd, H^{p^r} is powerfully embedded in G^{p^r}.
(b) If $p = 2$, $(H^{2^r})^2$ is powerfully embedded in $(G^{2^r})^2$.

Theorem 4.14. *Let G be a locally finite p-group and H be a normal right n-Engel subgroup of G. There exists an integer $s = s(n)$ such that the following hold.*
(a) If p is odd, $[H^{p^r},_s G^{p^r}] = 1$.
(b) If $p = 2$, $[(H^{2^r})^2,_s (G^{2^r})^2] = 1$.

As in the n-Engel group case, the r in Theorem 4.14 is close to being the best bound. Let t be the smallest positive integer such that H^{p^t} is upper central of n-bounded degree in G^{p^t}. By the example given in Section 3.5, $t \in \{r-1, r\}$ if p is odd and $t \in \{r-1, r, r+1\}$ if $p = 2$. We now prove the above theorems. We use a similar method to the proofs of Theorems 3.9 - 3.11 as proved in [1].

Proof of Theorem 4.12. Let G be a finite p-group and suppose that H is a right n-Engel subgroup which is powerfully embedded in G. As H is powerfully embedded in G we have that H^{p^k} is powerfully embedded in G for any positive integer k. Furthermore $H^{p^k} = \{h^{p^k} : h \in H\}$. We use these properties to show by induction on $k \geq 1$ that $[H,_k G] \leq H^{p^k}$. The induction basis is given by the assumption that H is powerfully embedded in G. Now suppose that $k \geq 2$ and that the result holds for smaller values than k. Then

$$[H,_k G] = [H,_{k-1} G, G] \leq [H^{p^{k-1}}, G] \leq (H^{p^{k-1}})^p = H^{p^k}.$$

This finishes the inductive proof. By Theorem 4.3 there exist integers $c = c(n)$ and $e = e(n)$ such that $H^e \leq Z_c(G)$. Let $v = v(n)$ be the largest power of any prime that occurs in $e(n)$. Then $[H^{p^v},_c G] = \{1\}$ and thus $[H,_{v+c} G] \leq [H^{p^v},_c G] = \{1\}$. Taking $s(n) = v(n) + c(n)$ gives the result. $\square$

We now turn to the proofs of Theorems 4.13 and 4.14. Let p be a fixed prime and n be a fixed positive integer. Let r be the integer satisfying $p^{r-1} < n \leq p^r$.

Proof of Theorem 4.13. (a) We can assume that $(H^{p^r})^p = \{1\}$ and then the aim is to show that $[H^{p^r}, G^{p^r}] = \{1\}$. Let $g \in G$ be arbitrary and set $V = H^{p^r}$. Since H is a finite n-Engel p-group, we have by Theorem 3.10 that V is powerful and hence elementary abelian. Since H is a right n-Engel subgroup, for each $v \in V$, $[v,_n g] = 1$ and thus $[v,_{p^r} g] = 1$. Hence, in End(V), $0 = (-1+g)^{p^r} = g^{p^r} - 1$. So $[v, g^{p^r}] = 1$ as required.

(b) Let $K = (H^{2^r})^2$. We may assume that $K^4 = \{1\}$ and then the aim is to show that $[K, (G^{2^r})^2] = \{1\}$. Let $g \in G$ be arbitrary and set $V = K/K^2$. As H is a right n-Engel subgroup, we have by Theorem 3.10 that K is powerful and hence abelian. It follows that V is an elementary abelian 2-group and, for each $v \in V$, $[v,_{2^r} g] = 1$. We can conclude that in $\mathrm{End}(V)$, $0 = (-1+g)^{2^r} = g^{2^r} - 1$. This shows that $[K, G^{2^r}] \leq K^2$. Let $k \in K$, then

$$[k, (g^{2^r})^2] = [k, g^{2^r}]^2 [k, g^{2^r}, g^{2^r}]$$

and since $[k, g^{2^r}] \in K^2$ and K is abelian we have that $[k, g^{2^r}]^2 \in K^4 = \{1\}$. It remains to see that $[k, g^{2^r}, g^{2^r}] = 1$ and for this it suffices to show that $[K^2, G^{2^r}] = \{1\}$. But as K^2 is right n-Engel and elementary abelian we have as before that in $\mathrm{End}(K^2)$, $0 = (-1+g)^{2^r} = g^{2^r} - 1$, and so $[K^2, G^{2^r}] = \{1\}$. This finishes the proof. $\Box$

Proof of Theorem 4.14. (a) Let $s = s(n)$ be as in Theorem 4.12. Let $h \in H$ and $g_1, \ldots, g_s \in G^{p^r}$. Then $g_1, \ldots, g_s \in M^{p^r}$ for some finitely generated subgroup M of G. Let $K = \langle h, M \rangle$. Since G is locally finite it follows that K is finite. By Theorem 4.13, $(\langle h \rangle^K)^{p^r}$ is powerfully embedded in K^{p^r}. Hence, by Theorem 4.12, $[h^{p^r}, g_1, \ldots, g_s] = 1$. This finishes the proof of part (a). Part (b) is proved similarly. $\Box$

Chapter 5

Right 2-Engel Subgroups

5.1 Introduction

We would like to find the best possible values for the integers appearing in the theorems in Chapter 4. Note that right 1-Engel subgroups are exactly subgroups lying in the centre of a group. Hence when $n = 1$ the best possible values are all one, except for r in Theorem 4.14, which can be taken to be zero. In this chapter we consider the simplest non-trivial case of right 2-Engel subgroups. The examples in this chapter appear in [10, 11].

5.2 Right 2-Engel subgroups

We first consider the values of the functions defined in Definition 4.5 for $n = 2$. To help find these we have the analogous version of Remark 3.8.

Remark 5.1. (i) Each $\hat{c}_1(d, n)$, $\hat{c}_i(n)$, $\hat{d}_i(n)$, $\hat{e}_i(n)$ and $\hat{f}_i(n)$ is an increasing function in n.
(ii) $\hat{c}_2(n) = \hat{c}_3(n) = \hat{c}_4(n)$ for a similar reason to Remark 3.8 (ii).
(iii) Note that if $[h^e, g_1, \ldots, g_c] = 1$, then $[h, g_1, \ldots, g_c]^e \in [H,_{c+1} G]$. One can then see from the proofs of Theorems 4.3 and 4.4 that, for a particular n, the prime divisors of both $\hat{f}_3(n)$ and $\hat{f}_4(n)$ are the same.
(iv) For a particular n, the prime divisors of both $\hat{e}_3(n)$ and $\hat{e}_4(n)$ are the same, which again can be seen from the proofs of Theorems 4.3 and 4.4.
(v) $\hat{e}_3(n)$ and $\hat{e}_4(n)$ divide $\hat{f}_3(n)$ and $\hat{f}_4(n)$ respectively.

Let G be any group, $x, y, z \in G$ and h be a right 2-Engel element of G. In [24, 27] (see also [35] Theorem 7.13), it is shown that $[h, x, y, z]^2 = 1$ and that $\langle h \rangle^G$ is an abelian right 2-Engel subgroup. Further, it follows that

$$[h^2, x, y, z] = 1.$$

We show that this is the best result one can obtain and hence that $\hat{c}_3(2) = 3$ and $\hat{e}_3(2) = 2$. From this it follows, by Remark 5.1, that $\hat{c}_2(2) = \hat{c}_3(2) = \hat{c}_4(2) = \hat{d}_3(2) = 3$ and $\hat{e}_3(2) = \hat{f}_3(2) = 2$. First we give an example to show that $\hat{c}_3(2) = 3$.

Example 1. Let M be the multiplicative group generated by the real matrices X and Y, where

$$X = \begin{pmatrix} 0 & -1 & -2 & -2 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix},$$

and let $N = \mathbb{R}^4$ considered as a group with respect to addition. Consider the semidirect product $G = N \rtimes M$, where M acts on N by matrix multiplication from the left. Let $v = (v_1, v_2, v_3, v_4) \in N$ and $w = v_1 + v_2 + v_3 + v_4$. Then,

$$[v, X, Y] = (w, -w, -w, w) = -[v, Y, X].$$

Direct calculations also show that $[v, X, X] = [v, Y, Y] = 0$. From these it follows that N is upper central in G of degree at most 3. Also, if $W = X^i Y^j U$, where $U \in M'$, then

$$[v, W, W] = [v, X^iY^j, X^iY^j] = i^2[v, X, X] + j^2[v, Y, Y] + ij([v, X, Y] + [v, Y, X]) = 0.$$

Hence v is a right 2-Engel element of G and N is a right 2-Engel subgroup of G. However, if v is chosen such that $w \neq 0$, then $[nv, X, Y] \neq 0$ and thus $nv \notin Z_2(G)$ for any $n \in \mathbb{N}$.

Now we give an example which shows that $\hat{e}_3(2) \geq 2$. We give an example of a group with a normal right 2-Engel subgroup that is upper central, but has arbitrary upper central degree.

Example 2. Let $n \in \mathbb{N}$ and consider $G = C_2 \operatorname{wr} C_2^n = \prod_{g \in C_2^n} C_2^g \rtimes C_2^n$. Let $B = \prod_{g \in C_2^n} C_2^g$ be the base group and $M = C_2^n$ be the group acting on it. For each $g, h \in B$ and $x \in M$, we have

$$[g, hx, hx] = [g, x, x] = g^{(-1+x)^2} = g^{1-2x+x^2} = 1.$$

This shows that B is a right 2-Engel subgroup of G. Note that, as G is finite, B is upper central. However, if we let $e_i \in M$ be the element with (-1) in coordinate i, but 1 in all the other coordinates and we consider $(-1)^1 \in B$, then $[(-1)^1, e_1, e_2, \ldots, e_n] = (-1)^{(-1+e_1)\cdots(-1+e_n)} \neq 1$. This is non-trivial, as the C_2^1 part is, for example. Hence $[B,_n G] \neq \{1\}$.

Now we turn to the values arising from Theorem 4.4. We have seen that $\hat{c}_4(2) = 3$. Let h be a right 2-Engel element of a group G and $x, y, z \in G$. As mentioned above, $[h, x, y, z]^2 = 1$ and $\langle h\rangle^G$ is an abelian right 2-Engel subgroup. Hence we also have

$$1 = [h, x, xy, xy] = [h, x, y, xy] = [h, x, y, x]^y$$

and $[h, x^{-1}] = [h, x]^{-1}$. From this it is clear that any commutator $[h, u_1, \ldots, u_m]$ with $u_1, \ldots, u_m \in \{x, y\}$ and with a repeated entry of either x or y is trivial. In particular, such a commutator is trivial if $m \geq 3$. It follows that

$$1 = [h, xy, xy] = [h, x, y][h, y, x]$$

and $[h, y, x] = [h, x, y]^{-1}$. It follows that if $h \in H$ and $x_1, \ldots, x_m \in G$, then any commutator $[h, x_{i_1}, \ldots, x_{i_t}]$, with some x_i repeated, is trivial. Thus for $h \in H$ and $x, y, z \in G$, we have

$$[h, x, [y, z]] = [h, x, y, z][h, x, z, y]^{-1} = [h, x, y, z]^2 = 1.$$

It follows that $[H, G, G, G] = \langle [h, x, y, z] : h \in H, x, y, z \in G\rangle$ is abelian and so $[H, G, G, G]^2 = 1$. Thus Example 2 shows that $\hat{d}_4(2) = 3$ and $\hat{e}_4(2) = \hat{f}_4(2) = 2$.

Having sorted out the integers in Theorems 4.2 - 4.4, we determine the integers $\hat{c}_1(d, 2)$, arising from Theorem 4.1. Trivially, $\hat{c}_1(1, n) = 1$, for all $n \geq 1$. We have seen above that any commutator with first entry h and two identical entries is trivial. Thus if $G = \langle g_1, \ldots, g_d\rangle$, then any commutator $[h, g_{i_1}, \ldots, g_{i_{d+1}}] = 1$. So $h \in Z_{d+1}(G)$. Hence $\hat{c}_1(d, 2) \leq d+1$ and every right 2-Engel subgroup of a finitely generated group is upper central. The following example shows that $\hat{c}_1(d, 2) = d + 1$ when $d \geq 2$.

Example 3. Let $d \geq 2$. Let F be the free associative algebra with unity over GF(2), the field with two elements, generated by $x_1, x_2, \dots, x_d$. Let I be the ideal generated by all monomials of multiweight $(w_1, w_2, w_3, \dots, w_d)$ in $x_1, \dots, x_d$, where either one of w_1, w_2 is at least three or one of $w_3, w_4, \dots, w_d$ is at least two. Let J be the ideal generated by all elements of the form

$$x_i x_j x_{i_1} \cdots x_{i_r} - x_i x_j x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(r)}}$$

with $r \geq 2$, $i, j, i_1, \dots, i_r \geq 1$ and $\sigma \in S_r$. Consider the ring $R = F/(I+J)$. Let $y_i = x_i + (I+J)$. Notice that, by the definition of J, any $y_i y_j y_{i_1} \cdots y_{i_r}$ is symmetric in the last r factors. Let $R_{(w_1, w_2, \dots, w_d)}$ be the subspace generated by all monomials of multiweight $(w_1, w_2, \dots, w_d)$ in $y_1, y_2, \dots, y_d$. Then

$$R = \sum_{(w_1, \dots, w_d) \in \mathbb{N}^d} R_{(w_1, \dots, w_d)},$$

where $R_{(w_1, \dots, w_d)} = \{0\}$ whenever either one of w_1, w_2 is at least 3 or one of $w_3, \dots, w_d$ is at least 2, by the definition of I.

Let S be the ideal generated by all monomials of multiweight greater than $(0, \dots, 0)$ in $(y_1, \dots, y_d)$. Then $E = 1 + S$ is a finite 2-group with respect to multiplication, since, for every $s \in S$, $(1+s)^{2^{d+1}} = 1 + s^{2^{d+1}} = 1$, as monomials of degree 2^{d+1} are trivial. We can consider S as a Lie ring with respect to the bracket product $(s_1, s_2) = s_1 s_2 - s_2 s_1 = s_1 s_2 + s_2 s_1$. We will use the well known fact that if s_1, s_2 are multihomogenous elements in $R_{(\alpha_1, \dots, \alpha_d)}$ and $R_{(\beta_1, \dots, \beta_d)}$ respectively, then $[1+s_1, 1+s_2] = 1 + (s_1, s_2) + t$, where t is a sum of monomials of higher multiweight than $(\alpha_1 + \beta_1, \dots, \alpha_d + \beta_d)$ in $(y_1, \dots, y_d)$. This can be easily verified by expanding $(1+s_1)^{2^{d+1}-1}(1+s_2)^{2^{d+1}-1}(1+s_1)(1+s_2)$.

Let $b_i = 1 + y_i$ and let G be the subgroup of E generated by $b_1, \dots, b_d$. We consider the element $a = [b_1, b_2] = 1 + u$. Calculations give that

$$\begin{aligned} u =& (1+y_1)^3(1+y_2)^3(1+y_1)(1+y_2) - 1 \\ =& (y_1, y_2) + (y_1^2 y_2 + y_1 y_2 y_1) + (y_2^2 y_1 + y_2 y_1 y_2) + (y_1 y_2^2 y_1 + y_1 y_2 y_1 y_2). \end{aligned}$$

Note that

$$u^2 = y_1 y_2 y_1 y_2 + y_1 y_2 y_2 y_1 + y_2 y_1 y_1 y_2 + y_2 y_1 y_2 y_1 = 2(y_1 y_2 y_1 y_2 + y_2 y_1 y_1 y_2) = 0.$$

Also, since we have quotiented out by the ideal J we have, for all $s_1, s_2 \in S$,

$$u(s_1s_2 + s_2s_1) = 2us_1s_2 = 0, \tag{5.1}$$
$$s_1s_2u = 2s_1s_2(y_1y_2 + y_1^2y_2 + y_1y_2^2 + y_1^2y_2^2) = 0. \tag{5.2}$$

We now show that a is a right 2-Engel element of G. Notice first that as we have quotiented out by the ideal I we have

$$[a, b_{i_1}, \ldots, b_{i_r}] = 1 + (u, y_{i_1}, \ldots, y_{i_r}), \tag{5.3}$$

as any monomial in S of higher multiweight than $(1, 1, \ldots, 1)$ in $u, y_{i_1}, \ldots, y_{i_r}$ is zero. For the same reason, (5.3) is trivial if there is a repetition among $y_{i_1}, \ldots, y_{i_r}$. Using equation (5.2) we see that

$$(u, y_{i_1}, \ldots, y_{i_r}) = u \prod_{j=1}^{r} y_{i_j} + \sum_{j=1}^{r} y_{i_j} u \prod_{k \neq j} y_{i_k}.$$

This last expression is symmetrical in $y_{i_1}, \ldots, y_{i_r}$. For a subset $I = \{j_1, \ldots, j_s\}$ of $\{1, \ldots, r\}$ and $b = 1 + v \in \langle a \rangle^G$, we will use the notation (v, y_I) for $(v, y_{i_{j_1}}, \ldots, y_{i_{j_s}})$ and similarly $[b, b_I]$ for the commutator expression $[b, b_{i_{j_1}}, \ldots, b_{i_{j_s}}]$.

Now take any $g = b_{i_1} \cdots b_{i_r} \in G$. From the discussion above we see that

$$\begin{aligned}
[a, g, g] &= \prod_{\emptyset \neq I, J \subseteq \{1,\ldots,r\}} [a, b_I, b_J] \\
&= \prod_{\emptyset \neq I, J \subseteq \{1,\ldots,r\}} 1 + (u, y_I, y_J) \\
&= 1 + \sum_{\emptyset \neq I, J \subseteq \{1,\ldots,r\}} (u, y_I, y_J), \text{ by (5.1)}.
\end{aligned}$$

Now, $(u, y_I, y_J) = 0$ if $I \cap J \neq \emptyset$. As $(u, y_I, y_J) = (u, y_J, y_I)$, the remaining terms come in pairs. Since R is of characteristic 2, it follows that $[a, g, g] = 1$. Thus a is a right 2-Engel element of G and thus $H = \langle a \rangle^G$ is a right 2-Engel subgroup. However,

$$[a, b_1, b_2, b_3, \ldots, b_d] = 1 + (y_1, y_2, y_1, y_2, y_3, \ldots, y_d).$$

This commutator is non-trivial since, for example, after expansion the coefficient of the basis element $y_1^2y_2^2y_3 \cdots y_d$ is 1. Thus H is not in $Z_d(G)$.

5.3 Right 2-Engel subgroups of p-groups

In this section we consider Theorems 4.12 and 4.14 for the case $n = 2$. Let $\hat{s}(2, p)$ be the smallest positive integer such that $[H,_{\hat{s}(2,p)} G] = \{1\}$ for all pairs (H, G), where G is a finite p-group and H is a right 2-Engel subgroup that is powerfully embedded in G. Also, let $\hat{c}(2, p)$ and $\hat{f}(2, p)$ be integers such that for any pair (H, G), where G is a locally finite p-group and H is a normal right 2-Engel subgroup of G,

$$[H^{p^{\hat{f}(2,p)}},_{\hat{c}(2,p)} G^{p^{\hat{f}(2,p)}}] = \{1\}.$$

We want to find the value of $\hat{s}(2, p)$ and the best possible values for $\hat{c}(2, p)$ and $\hat{f}(2, p)$.

First we deal with the case p is odd. We have seen in Section 5.2 that $[h^2, x, y, z] = 1$, when h is a right 2-Engel element and $x, y, z \in G$. Hence $[h, x, y, z] = 1$ when p is odd. It follows that $\hat{s}(2, p) \leq 3$ and that the best possible value for $\hat{f}(2, p)$ is 0. The next example shows that $\hat{s}(2, p) = 3$ and that the best possible value for $\hat{c}(2, p)$ is 3.

Example 4. For any given positive integer s we let $\mathbb{Z}_{p^s}$ be the congruence class of the integers modulo p^s. We let $N(s) = \mathbb{Z}_{p^s}^4$ and $M(s)$ be the subgroup of $\mathrm{GL}(4, \mathbb{Z}_{p^s})$, generated by

$$X(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -p & 1 \end{pmatrix}, \quad Y(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 1 & 0 \\ 0 & p & 0 & 1 \end{pmatrix}.$$

Let $L(s) = N(s) \rtimes M(s)$ where $M(s)$ acts on $N(s)$ by multiplication on the left. Notice that if v_1, v_2, v_3, v_4 is the standard basis for $N(s)$, then

$$\begin{array}{llll} [v_1, X] = pv_2, & [v_2, X] = 0, & [v_3, X] = -pv_4, & [v_4, X] = 0, \\ [v_1, Y] = pv_3, & [v_2, Y] = pv_4, & [v_3, Y] = 0, & [v_4, Y] = 0. \end{array}$$

Notice that $L(s)$ is a finite p-group and that $N(s)$ is powerfully embedded in $L(s)$. Further, notice that for $v \in \{v_1, v_2, v_3, v_4\}$,

$$[v, X, X] = [v, Y, Y] = [v, X, Y, X] = [v, Y, X, Y] = 0$$

and

$$[v, X, Y] = -[v, Y, X].$$

Hence, if $u \in M(s)'$ and $i, j \in \mathbb{Z}$, then

$$[v, X^i Y^j u, X^i Y^j u] = [v, X^i Y^j, X^i Y^j] = ij[v, X, Y] + ij[v, Y, X] = 0.$$

Thus $N(s)$ is a right 2-Engel subgroup of $L(s)$. Also, for $t \in \mathbb{N}$,

$$[p^t v_1, X^{p^t}, Y^{p^t}] = p^{3t}[v_1, X, Y] = p^{3t+2} v_4,$$

which is non trivial in $L(3t+3)$. This shows that the best possible value for $\hat{c}(2,p)$ is 3. Since $[v_1, X, Y] = p^2 v_4$ is non-trivial in $L(3)$, we also see that $\hat{s}(2,p) = 3$.

It remains to deal with $p = 2$. We know that, for $h \in H$ and $x, y, z \in G$, $[h^2, x, y, z] = 1$ and hence $[h^2, x^2, y^2, z^2] = 1$. Thus the best value for $\hat{c}(2,2)$ is at most 3 and the best value for $\hat{f}(2,2)$ is at most 1. Notice that Example 4 holds for $p = 2$, except that $N(2)$ is not powerfully embedded in $L(2)$. Thus the best possible value for $\hat{c}(2,2)$ is 3 and it remains to deal with $\hat{s}(2,2)$ and $\hat{f}(2,2)$. Example 2 from Section 5.2 shows that the best possible value of $\hat{f}(2,2)$ is 1.

It now only remains to deal with $\hat{s}(2,2)$. As we have remarked before, we know that $[H^2,_3 G] = \{1\}$ for any pair (H, G) where H is a normal right 2-Engel subgroup of G. Thus, if H is a powerfully embedded subgroup of a finite 2-group G, then

$$[H,_4 G] \leq [H^4,_3 G] = \{1\}.$$

We now show that $\hat{s}(2,2) = 4$, by giving an example that shows that $\hat{s}(2,2) > 3$.

Example 5. The construction is similar to the one in Example 4. This time we let $N(s) = \mathbb{Z}_{2^s}^8$ and we let $M(s)$ be the subgroup of $\mathrm{GL}\,(8, \mathbb{Z}_{2^s})$ generated by the following three matrices

$$X(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \end{pmatrix}, Y(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 1 \end{pmatrix}$$

and

$$Z(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 1 \end{pmatrix}.$$

We then let $L(s) = N(s) \rtimes M(s)$, where as before $M(s)$ acts on $N(s)$ by matrix multiplication on the left. If we let $v_1, \ldots, v_8$ be the standard basis for $N(s)$, then

$$\begin{array}{llll} [v_1, X] = 4v_2, & [v_2, X] = 0, & [v_3, X] = -4v_5, & [v_4, X] = -4v_6, \\ [v_5, X] = 0, & [v_6, X] = 0, & [v_7, X] = 4v_8, & [v_8, X] = 0, \\ [v_1, Y] = 4v_3, & [v_2, Y] = 4v_5, & [v_3, Y] = 0, & [v_4, Y] = -4v_7, \\ [v_5, Y] = 0, & [v_6, Y] = -4v_8, & [v_7, Y] = 0, & [v_8, Y] = 0, \\ [v_1, Z] = 4v_4, & [v_2, Z] = 4v_6, & [v_3, Z] = 4v_7, & [v_4, Z] = 0, \\ [v_5, Z] = 4v_8, & [v_6, Z] = 0, & [v_7, Z] = 0, & [v_8, Z] = 0. \end{array}$$

Note that the group $L(s)$ is a finite 2-group where $N(s)$ is powerfully embedded in $L(s)$. Notice that for all $v \in \{v_1, \ldots, v_8\}$,

$$[v, X, X] = [v, Y, Y] = [v, Z, Z] = 0$$

and

$$[v, X, Y] = -[v, Y, X],\ [v, X, Z] = -[v, Z, X],\ [v, Y, Z] = -[v, Z, Y].$$

Hence $[N(s),_4 L(s)] = \{0\}$ and

$$\begin{aligned} &[v, [X, Y], Z] = [v, [Y, Z], X] = [v, [Z, X], Y] \\ =&[v, X, [Y, Z]] = [v, Y, [Z, X]] = [v, Z, [X, Y]] = 2[v, X, Y, Z]. \end{aligned}$$

In $L(7)$ we have $2[v, X, Y, Z] \in 2^7 N(s) = \{0\}$, which implies that the above commutators are trivial. Hence in $L(7)$, for $i, j, k \in \mathbb{Z}$ and $u \in L(7)'$,

$$\begin{aligned}
[v, X^iY^jZ^ku, X^iY^jZ^ku] =& [v, X^iY^jZ^k, X^iY^jZ^k] \\
=& ij([v, X, Y] + [v, Y, X]) + ik([v, X, Z] + [v, Z, X]) \\
& + jk([v, Y, Z] + [v, Z, Y]) + ijk([v, X, Y, Z] + [v, Y, Z, X]) \\
& + ijk([v, Y, X, Z] + [v, X, Z, Y] + [v, Z, X, Y] + [v, X, Y, Z]) \\
=& ijk(2[v, X, Y, Z]) = 0.
\end{aligned}$$

Hence $N(7)$ is a right 2-Engel subgroup of $L(7)$. However,

$$[v_1, X, Y, Z] = 2^6 v_8 \neq 0.$$

This shows that $\hat{s}(2, 2) = 4$.

Chapter 6

Right 3-Engel Subgroups

6.1 Introduction

In this chapter we derive some general structure results for right 3-Engel subgroups. Kappe and Kappe [25] have shown that every 3-Engel group is Fitting, with Fitting degree at most 2. That is, the normal closure of any element is nilpotent, with nilpotency class at most 2. This is equivalent to the statement that every commutator in a 3-Engel group with a triple entry, i.e. three identical entries, is trivial. Newell [30] has shown that in any group the normal closure of a right 3-Engel element is nilpotent of class at most 3. We will prove a related result about normal right 3-Engel subgroups. For this we will need to impose some extra conditions on the subgroup. We will need it to be both upper central and 3-torsion-free. We will prove that, with these conditions, every commutator with a quadruple entry and an entry from the right 3-Engel subgroup is trivial. We then show that this cannot be improved.

Using this result we will then be able to bound $\hat{c}_2(3)$ from above. Recall that $\hat{c}_2(3)$ is the smallest positive integer such that every torsion-free normal upper central right 3-Engel subgroup H of an arbitrary group G has the property $H \leq Z_{\hat{c}_2(3)}G$. In fact the exceptional primes here are 2, 3 and 5. The primes 2 and 5 are exceptional for 3-Engel groups and hence are exceptional for right 3-Engel subgroups too. We will see that the prime 3 is exceptional in an example at the end of Section 6.3.

We will then bound $\hat{c}_2(3)$ from below to show that $\hat{c}_2(3) = 8$. For this we give a counterexample to $\hat{c}_2(3)$ being less than 8. This will involve lengthy calculations which we spend some time shortening beforehand. From this result it follows that $\hat{c}_3(3)$ and $\hat{c}_4(3)$ are both 8 as well. The fact that 2, 3 and 5 are exceptional shows that these are the prime divisors of $\hat{e}_3(3)$, $\hat{e}_4(3)$, $\hat{f}_3(3)$ and $\hat{f}_4(3)$.

6.2 Fitting result

We start with some identities for normal right 3-Engel subgroups, which will be useful throughout this chapter. We consider commutators with an entry set of three elements, one of which is from a normal right 3-Engel subgroup. We will assume that this subgroup is abelian in order to simplify matters. This condition will be removed later. We need not impose that the subgroup is 3-torsion-free for the first two identities.

Lemma 6.1. *Let H be a normal right 3-Engel subgroup of a group G. Suppose also that H is abelian. Let $h \in H$ and $x, y \in G$. Then*
(i) $[h,x,x,y][h,x,y,x][h,y,x,x][h,x,y,y][h,y,x,y][h,y,y,x][h,y,x,y,x]$
$[h,x,x,y,y]^{-1} = 1$.
(ii) $[h,y,x,y,x]^{-1}[h,y,y,x,x]^{-1}[h,x,y,x,y][h,x,x,y,y] = 1$.

Proof. Since H is abelian, we can work in the endomorphism ring of H. For $g \in G$ we define $R_g : H \mapsto H$ by $R_g(h) = [h,g]$, which is possible as H is normal. Note that for $h_1, h_2 \in H$, $[h_1h_2, g] = [h_1, g][h_2, g]$ and thus R_g is an endomorphism. We also have

$$R_{xy} = R_x + R_y + R_xR_y = R_x + (1 + R_x)R_y = R_x(1 + R_y) + R_y.$$

Thus, since $R_x^3 = R_y^3 = 0$,

$$\begin{aligned}(1 - R_x + R_x^2)R_{xy} &= R_x - R_x^2 + R_y,\\ R_{xy}(1 - R_y + R_y^2) &= R_x + R_y - R_y^2.\end{aligned}$$

Hence,

$$\begin{aligned}0 &= (1 - R_x + R_x^2)R_{xy}^3(1 - R_y + R_y^2)\\ &= (R_x - R_x^2 + R_y)(R_x + R_y + R_xR_y)(R_x + R_y - R_y^2)\\ &= (R_x + R_y)^3 - R_x^2R_y(R_x + R_y) + (R_x + R_y)R_xR_y(R_x + R_y) - (R_x + R_y)R_xR_y^2\\ &= (R_x + R_y)^3 + R_yR_xR_y(R_x + R_y) - (R_x + R_y)R_xR_y^2\\ &= (R_x + R_y)^3 + R_yR_xR_yR_x - R_x^2R_y^2.\end{aligned}$$

Expanding and applying to h gives (i). Swapping the roles of x and y in (i) and multipling by the inverse of (i) gives (ii). □

We now consider 3-torsion-free normal right 3-Engel subgroups. We will see later that it is necessary for the main results we are proving that the subgroup is 3-torsion-free. We use Lemma 6.1(i) to give us some more identities. Here we slightly weaken the assumption that the subgroup is abelian.

Lemma 6.2. *Let H be a 3-torsion-free normal right 3-Engel subgroup. Let $h \in H$ and $x, y \in G$. Suppose that commutators of multiweight at least (2,3,1) in h, x, y are trivial. Then*
(i) $[h, x, x, y, x, x] = 1$.
(ii) $[h, x, x, y, x][h, x, y, x, x] = 1$.
(iii) $[h, x, y, x, y, x] = 1$.

Proof. From Lemma 6.1(i) we may assume that

$$\begin{aligned}&[h,x,x,y][h,x,y,x][h,y,x,x][h,x,y,y][h,y,x,y][h,y,y,x]\\&[h,y,x,y,x][h,x,x,y,y]^{-1}uv=1,\end{aligned} \tag{6.1}$$

where u is a product of commutators of multiweight at least (2,1,1) in h, x, y with one entry of x and v is a product of commutators of multiweight at least (2,2,1) in h, x, y. Let $n \in \mathbb{N}$. Now,

$$[h, x^n] = [h, x][h, x^{n-1}][h, x, x]^{n-1}.$$

Thus by induction we have

$$[h, x^n] = [h, x]^n [h, x, x]^{\binom{n}{2}}.$$

Hence replacing x by x^n in (6.1) gives

$$\begin{aligned}1 =&[h,x,x,y]^{n^2}[h,x,y,x]^{n^2}[h,x,x,y,x]^{n\binom{n}{2}}[h,x,y,x,x]^{n\binom{n}{2}}[h,x,x,y,x,x]^{\binom{n}{2}^2}\\&[h,y,x,x]^{n^2}[h,x,y,y]^{n}[h,x,x,y,y]^{\binom{n}{2}}[h,y,x,y]^{n}[h,y,x,x,y]^{\binom{n}{2}}[h,y,y,x]^{n}\\&[h,y,y,x,x]^{\binom{n}{2}}[h,y,x,y,x]^{n^2}[h,y,x,x,y,x]^{n\binom{n}{2}}[h,y,x,y,x,x]^{n\binom{n}{2}}\\&[h,y,x,x,y,x,x]^{\binom{n}{2}^2}[h,x,x,y,y]^{-n^2}u^n v^{n^2} w^{\binom{n}{2}},\end{aligned} \tag{6.2}$$

where w is a product of commutators of multiweight at least (2,2,1) in h, x, y. Note that by the Hall-Petrescu formula (Theorem 2.7),

$$\begin{aligned}[h,x,y,y]^n[h,y,x,y]^n[h,y,y,x]^n u^n =&[h,x,y,y]^n[h,y,x,y]^n([h,y,y,x]u)^n a^{\binom{n}{2}}\\=&([h,x,y,y][h,y,x,y][h,y,y,x]u)^n b^{\binom{n}{2}},\end{aligned}$$

where a and b are products of commutators of multiweight at least (2,2,1) in h, x, y.

Multiplying (6.2) by (6.1) raised to the power of $-n$ and using the above gives

$$\begin{aligned}1 =&[h,x,x,y]^{2\binom{n}{2}}[h,x,y,x]^{2\binom{n}{2}}[h,y,x,x]^{2\binom{n}{2}}[h,x,x,y,x]^{n\binom{n}{2}}[h,x,y,x,x]^{n\binom{n}{2}}\\&[h,x,x,y,y]^{-\binom{n}{2}}[h,y,x,x,y]^{\binom{n}{2}}[h,y,y,x,x]^{\binom{n}{2}}[h,y,x,y,x]^{2\binom{n}{2}}[h,y,x,x,y,x]^{n\binom{n}{2}}\\&[h,y,x,y,x,x]^{n\binom{n}{2}}[h,x,x,y,x,x]^{\binom{n}{2}^2}[h,y,x,x,y,x,x]^{\binom{n}{2}^2}v^{2\binom{n}{2}}t^{\binom{n}{2}},\end{aligned}$$

where t is a product of commutators of multiweight at least (2,2,1) in h, x, y. Setting $n = 2$ gives

$$\begin{aligned}1 =&[h,x,x,y]^2[h,x,y,x]^2[h,y,x,x]^2[h,x,x,y,x]^2[h,x,y,x,x]^2[h,x,x,y,y]^{-1}\\&[h,y,x,x,y][h,y,y,x,x][h,y,x,y,x]^2[h,y,x,x,y,x]^2[h,y,x,y,x,x]^2\\&[h,x,x,y,x,x][h,y,x,x,y,x,x]v^2t. \qquad (6.3)\end{aligned}$$

Multiplying this to the power of $-\binom{n}{2}$ by the previous equation gives

$$\begin{aligned}1 =&[h,x,x,y,x]^{(n-2)\binom{n}{2}}[h,x,y,x,x]^{(n-2)\binom{n}{2}}[h,y,x,x,y,x]^{(n-2)\binom{n}{2}}\\&[h,y,x,y,x,x]^{(n-2)\binom{n}{2}}[h,x,x,y,x,x]^{\binom{n}{2}^2-\binom{n}{2}}[h,y,x,x,y,x,x]^{\binom{n}{2}^2-\binom{n}{2}}.\end{aligned}$$

Setting $n = 3$ gives

$$\begin{aligned}1 =&[h,x,x,y,x]^3[h,x,y,x,x]^3[h,y,x,x,y,x]^3\\&[h,y,x,y,x,x]^3[h,x,x,y,x,x]^6[h,y,x,x,y,x,x]^6. \qquad (6.4)\end{aligned}$$

Replacing h by $[h,y,y]$ and commuting by x gives $[h,y,y,x,x,y,x,x] = 1$, as H is 3-torsion-free. Replacing h by $[h,y,y]$ in (6.4) gives $[h,y,y,x,x,y,x][h,y,y,x,y,x,x] = 1$. Replacing h by $[h,y]$ in (6.4) gives

$$1 = [h,y,x,x,y,x]^3[h,y,x,y,x,x]^3[h,y,x,x,y,x,x]^6.$$

Hence (6.4) becomes

$$1 = [h,x,x,y,x]^3[h,x,y,x,x]^3[h,x,x,y,x,x]^6.$$

Commuting by x gives (i) and then (ii) follows. Replacing y by y^2 in (ii) gives

$$1 = [h,x,x,y,y,x][h,x,y,y,x,x].$$

Commuting Lemma 6.1(i) by x and replacing h by $[h, x]$ gives, using (i) and the above,

$$1 = [h, x, y, x, y, x][h, x, y, x, y, x, x].$$

Commuting by x gives $1 = [h, x, y, x, y, x, x]$ and hence we have (iii). $\square$

We now assume again that the subgroup is abelian and find three more identities.

Lemma 6.3. *Let H be a normal right 3-Engel subgroup of a group G. Suppose also that H is abelian and 3-torsion-free. Let $h \in H$ and $x, y \in G$. Then*
(i) $[h, y, x, x, y][h, y, x, y, x][h, y, y, x, x] = 1$.
(ii) $[h, x, x, y]^2[h, x, y, x]^2[h, y, x, x]^2[h, x, x, y, y]^{-1}[h, y, x, y, x] = 1$.
(iii) $[h, y, x, x, y][h, x, y, y, x]^{-1} = 1$.

Proof. Replacing h by $[h, y]$ in Lemma 6.1(i) and using Lemma 6.2(ii) gives

$$1 = [h, y, x, x, y][h, y, x, y, x][h, y, y, x, x][h, y, x, x, y, y]^{-1}[h, y, y, x, y, x].$$

Using Lemma 6.1(ii) and Lemma 6.2(iii) gives (i). Part (ii) then follows from (6.3), (i) and Lemma 6.2(i) and (ii), recalling that v and t were in H'. Commuting Lemma 6.1(i) by x gives

$$1 = [h, x, y, y, x][h, y, x, y, x][h, y, y, x, x][h, x, x, y, y, x]^{-1}[h, y, x, y, x, x]$$

Using Lemma 6.1(ii) and Lemma 6.2(iii) gives

$$1 = [h, x, y, y, x][h, y, x, y, x][h, y, y, x, x].$$

Together with (i) this gives (iii). $\square$

We can now use these identities to see that commutators with an entry of h and four x entries, two of which are consecutive, are trivial. We will also use the linearised 3-Engel identity, which holds for all right 3-Engel elements h in a group G. If $g_1, g_2, g_3 \in G$, then expanding $1 = [h, g_1g_2g_3, g_1g_2g_3, g_1g_2g_3]$ gives, modulo higher multiweights,

$$1 = \prod_{\sigma \in S_3} [h, g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}]. \tag{6.5}$$

Lemma 6.4. *Let H be a 3-torsion-free upper central normal right 3-Engel subgroup of a group G. Let $h \in H$ and $x \in G$. Suppose that commutators with at least two entries of h and four entries of x are trivial. Then any commutator with first entry h and at least four x entries, two of which are consecutive, is trivial.*

Proof. First suppose that there are two pairs of consecutive x entries. By Lemma 6.2(ii) commuted by x we have that $[h, x, x, y, x, x] = 1$. Replacing y by a product of arbitrarily many elements in G gives the result by induction. Thus we may suppose that there is only one pair of consecutive x entries. It then suffices to show that

$$[h, x, a, x, b, x, x] = [h, x, a, x, x, b, x] = [h, x, x, a, x, b, x] = 1.$$

By Lemma 6.2(ii) it suffices to see that $[h, x, a, x, b, x, x] = 1$. We will prove by induction on $m + n$ that a commutator

$$g = [h, x, a_1, \ldots, a_m, x, b_1, \ldots, b_n, x, x]$$

is trivial modulo higher multiweight commutators also with first entry h and two consecutive x entries, which will complete the proof since H is upper central. First we prove the base case $m = n = 1$ and assume that higher multiweight commutators with first entry h and two consecutive x entries are trivial.

Let $g_1 = a$, $g_2 = x$ and $g_3 = [x, b]$ in (6.5). Commuting twice by x we get

$$\begin{aligned}
1 =& [h, a, x, [x, b], x, x][h, x, a, [x, b], x, x][h, x, [x, b], a, x, x][h, [x, b], x, a, x, x] \\
=& [h, a, x, x, b, x, x][h, x, a, x, b, x, x][h, x, x, b, a, x, x][h, x, b, x, a, x, x]^{-1} \\
& [h, x, b, x, a, x, x][h, b, x, x, a, x, x]^{-1} \\
=& [h, x, a, x, b, x, x].
\end{aligned}$$

This establishes the base case. Suppose that $m + n > 2$ and the statement is true for values smaller than $m + n$. Assume that commutators of higher multiweight than g with first entry h and two consecutive x entries are trivial. It remains to see that $g = 1$.

Suppose that $m \geq 2$. Replacing a_i, a_{i+1} by the single element $[a_i, a_{i+1}]$ would make g trivial and hence

$$1 = [h, \ldots, [a_i, a_{i+1}], \ldots] = [h, \ldots, a_i, a_{i+1}, \ldots][h, \ldots, a_{i+1}, a_i, \ldots]^{-1}$$

Hence we may swap a_i and a_{i+1}. Thus we may arrange $a_1, \ldots, a_m$ and similarly $b_1, \ldots, b_n$ as we choose.

Let $a = a_1 \cdots a_m$ and $b = b_1 \cdots b_n$. Let $k = [h, x, a, x, b, x, x]$. Then, by inductive hypothesis, $1 = ku$, where u is a product of commutators of higher multiweight than k

in h, x, a, b, with weight at least that of g and with first entry h and at least two consecutive x entries. Suppose that u only consists of commutators with higher weight than g. Expanding $ku = 1$ in terms of $a_1, \ldots, a_m, b_1, \ldots, b_n$ and only considering commutators with at least one entry of each a_i and b_i would give $g = 1$. The commutators in u of the same weight as g can either be written as higher weight commutators by inductive hypothesis, or in the same form as g by Lemma 6.2(ii), with possibly different values for m and n and $a_1, \ldots, a_m, b_1, \ldots, b_n \in \{a, b\}$. From this it follows that we may assume that $a_1, \ldots, a_m, b_1, \ldots, b_n \in \{a, b\}$, where a and b are now arbitrary elements of G.

Suppose without loss of generality that there are at least two entries of a in g. If there are two entries of a within $a_1, \ldots, a_m$, then rearranging we get $[h, x, a, a, \ldots]$. By Lemma 6.1(i) and inductive hypothesis this is trivial. If there are two entries of a in $b_1, \ldots, b_n$, then by Lemma 6.1(i), inductive hypothesis and the case $a_1 = a_2 = a$,

$$1 = [h, \ldots, x, a, a, \ldots][h, \ldots, a, x, a, \ldots].$$

Thus we may assume that $a_1 = b_1 = a$. If $m = 1$, then the commutator is of the form $[h, x, a, x, a, \ldots]$. By Lemma 6.1(i) and the case $[h, x, a, a, \ldots]$ this is trivial. We are left with commutators of the form $[h, x, a, b, x, a, \ldots]$. By (6.5) with h replaced by $[h, x, a]$ and the case $a_1 = a_2 = a$, this is $[h, x, a, x, b, a, \ldots]^{-1}[h, x, a, x, a, b, \ldots]^{-1}$, which we have already seen to be trivial, as in both commutators $m = 1$. $\square$

We now remove the need for a pair of x entries to be consecutive.

Lemma 6.5. *Let H be a 3-torsion-free upper central normal right 3-Engel subgroup of a group G. Let $h \in H$ and $x \in G$. Suppose that commutators with at least two entries of h and four entries of x are trivial. Then any commutator with an entry of h and four x entries is trivial.*

Proof. We use a similar method to the proof of Lemma 6.4. By Lemma 6.4 it suffices to show that

$$g = [h, x, a_1, \ldots, a_l, x, b_1, \ldots, b_m, x, c_1, \ldots, c_n, x],$$

where $l, m, n \geq 1$ and $a_i, b_i, c_i \in G$, is trivial modulo commutators of higher multi-weight. We will use induction on $l + m + n$. First we prove the induction basis of $l + m + n = 3$. By (6.5) with $g_1 = a$, $g_2 = x$ and $g_3 = [x, b]$ and commuting by x, c, x, we get, modulo higher multiweights,

$$\begin{aligned}
1 =&[h,a,x,[x,b],x,c,x][h,a,[x,b],x,x,c,x][h,x,a,[x,b],x,c,x]\\
&[h,x,[x,b],a,x,c,x][h,[x,b],a,x,x,c,x][h,[x,b],x,a,x,c,x]\\
=&[h,x,a,x,b,x,c,x][h,x,b,x,a,x,c,x]^{-1}[h,x,b,x,a,x,c,x], \text{ by Lemma 6.4,}\\
=&[h,x,a,x,b,x,c,x].
\end{aligned}$$

Now suppose that $l+m+n>3$ and that the result holds for all smaller values than $l+m+n$. Suppose that all commutators of higher multiweight than g are trivial. If $l \geq 2$, replacing 'a_i, a_{i+1}' with $[a_i, a_{i+1}]$ would make g trivial and hence replacing this by 'a_i, a_{i+1}' and 'a_{i+1}, a_i' inversed shows that we may swap a_i and a_{i+1}. Thus we may arrange $a_1, \ldots, a_l$ and similarly $b_1, \ldots, b_m$ and $c_1, \ldots, c_n$ as we choose.

Let $a = a_1 \cdots a_l$, $b = b_1 \cdots b_m$, $c = c_1 \cdots c_n$ and $k = [h,x,a,x,b,x,c,x]$. By inductive hypothesis, $1 = ku$, where u is a product of commutators of higher multiweight than k in h,x,a,b,c and at least the weight of g. Suppose that u only consists of commutators with higher weight than g. Expanding $ku = 1$ in terms of $a_1, \ldots, a_l, b_1, \ldots, b_m, c_1, \ldots, c_n$ and only considering commutators with at least one entry of each a_i, b_i and c_i would give $g = 1$. The commutators in u of the same weight as g can either be written as higher weight commutators by inductive hypothesis, or in the same form as g, with possibly different values for l, m and n and $a_1, \ldots, a_l, b_1, \ldots, b_m, c_1, \ldots, c_n \in \{a,b,c\}$. Thus we may assume that each a_i, b_i and c_i is in $\{a,b,c\}$, where a, b and c are now arbitrary elements of G.

Since $l+m+n>3$, we may assume that there are at least two entries of a. Suppose that there are two a_i entries equal to a. Then rearranging we have $g = [h,x,a,a,\ldots]$, which is trivial by Lemma 6.1(i) and inductive hypothesis. Similarly, if there are two c_i entries equal to a, then $g = [\ldots,a,a,x]$ is trivial by Lemma 6.1(i) and inductive hypothesis. If $b_1 = b_2 = a$, then by Lemma 6.1(i) and the above, we have that $g = [\ldots,x,a,a,\ldots] = [\ldots,a,x,a,\ldots]^{-1}$. Hence we may assume that this is not the case.

Now suppose that $l = 1$ and $a_1 = a$. If $b_i = a$ for some i, then by rearranging we get $[h,x,a,x,a,\ldots]$, which is trivial by Lemma 6.1(i). If some $c_i = a$, then by (6.5),

$$1 = [h,x,a,x,\ldots,b_m,x,a,\ldots][h,x,a,x,\ldots,x,b_m,a,\ldots][h,x,a,x,\ldots,x,a,b_m,\ldots].$$

Using this repeatedly we end up with g as a product of commutators with the second and third x entries being consecutive. Hence g is trivial in this case. A similar argument applies to show that if $n = 1$ and $a_1 = c_1 = a$ or $b_1 = c_1 = a$, then $g = 1$.

Next suppose that $l > 1$ and $a_1 = b_1 = a$. Using (6.5) we have

$$1 = [h, x, a, \ldots, a_l, x, a, \ldots][h, x, a, \ldots, x, a_l, a, \ldots][h, x, a, \ldots, x, a, a_l, \ldots].$$

Using this repeatedly we end up with the case $l = 1$, which with these a entries we have seen is trivial. Similarly if $b_1 = c_1 = a$. If $a_1 = c_1 = a$, then again by (6.5) we have

$$1 = [h, x, a, \ldots, b_m, x, a, \ldots][h, x, a, \ldots, x, b_m, a, \ldots][h, x, a, \ldots, x, a, b_m, \ldots]$$

and using this repeatedly we have that g is a product of commutators where the second and third x entries are consecutive, which are trivial. Thus in every case $g = 1$, which completes the proof. □

It remains to remove the condition that commutators with at least two entries of h and four entries of x are trivial. We now do this, giving the first main result of the chapter.

Theorem 6.6. *Let H be a 3-torsion-free upper central normal right 3-Engel subgroup of a group G. Let $x \in G$. Then $[H,_4 \langle x \rangle^G] = 1$.*

Proof. Let $h \in H$. It suffices to see that any commutator with first entry h and four entries of x is trivial. Suppose that g is such a commutator, with other entries $a_1, \ldots, a_n$, $n \geq 0$. Let K be the subgroup of G generated by commutators with at least two entries of h and at least four entries of x. Quotienting G/K by every 3-element in H/K, which form a normal subgroup as H/K is nilpotent, allows us to apply Lemma 6.5. Hence $g^{3^\alpha} u = 1$ for some $\alpha \geq 0$ and $u \in K$. Independently setting each a_i to be trivial, we see that we may cancel any commutators in this product which do not contain at least one entry of each a_i. Thus we have that g^{3^α} is a product of commutators of higher multiweight, and as H is upper central and 3-torsion-free the result follows. □

6.3 Theorem 6.6 is best possible

In this section we check that Theorem 6.6 cannot be improved. We will show that under such conditions commutators with an entry from the right 3-Engel subgroup and a triple entry are not all trivial and that it was necessary to impose the 3-torsion-free condition. First we give an example satisfying the conditions of Theorem 6.6, but with $[H,_3 \langle x\rangle^G] \neq \{1\}$.

Let $G = \langle h, x, y\rangle$ be the relatively free group on these 3 generators, such that any commutator with at least two entries of h, two entries of y or four entries of x is trivial and $[h, x, x, x] = [h, y, x, x, x] = [h, x, x, y][h, x, y, x][h, y, x, x] = 1$. It follows that $[h, x, y, x, x][h, x, x, y, x] = 1$. From this definition it is clear that $H = \langle h\rangle^G$ is normal and upper central in G. By the discussion at the end of Section 2.3 there are no relations between left normed commutators in H with first entry h and entry set $\{h, x, y\}$, other than those which follow directly from the relations stated in the definition of G. One can check from these relations that H is 3-torsion-free. We claim that H is a right 3-Engel subgroup of G in which $[h, x, y, x, x] \neq 1$.

To show this, first note that H is a right 3-Engel subgroup of G if and only if

$$1 = [h, x^a y^b [x, y]^c, x^a y^b [x, y]^c, x^a y^b [x, y]^c],\ \forall a, b, c \in \mathbb{Z}.$$

Let $a, b, c \in \mathbb{Z}$ be arbitrary, $g = x^a y^b$ and $z = [x, y]^c$. Then, expanding the above, H is a right 3-Engel subgroup of G if and only if

$$1 = [h, g, g, g][h, g, g, z][h, g, z, g][h, z, g, g].$$

Now,

$$\begin{aligned}[h, g, g, z][h, g, z, g][h, z, g, g] =&([h, g, g, x, y][h, g, g, y, x]^{-1}[h, g, x, y, g]\\ &[h, g, y, x, g]^{-1}[h, x, y, g, g][h, y, x, g, g]^{-1})^c.\end{aligned}$$

Replacing both g's in each commutator with x^a makes the right hand side trivial, so it remains to see that $1 = [h, g, g, g]$. First consider when one of the g's is replaced by x^a, y^b. When the other two g's are replaced by x^a we get

$$([h, x, y, x, x][h, x, x, y, x][h, x, x, x, y])^{a^3 b},$$

which is trivial in G.

For $s \in \mathbb{Z}$, let $f(s) = \binom{s}{2} = s(s-1)/2$. We next consider when one g is replaced by x, x, which happens $f(a)$ ways. When the other g's are both replaced by x^a the product is clearly trivial. Replacing one g by x^a and the other by y^b gives

$$([h,x,x,x,y]^2[h,x,x,y,x][h,x,y,x,x][h,y,x,x,x]^2)^{f(a)ab} = 1.$$

Finally in the expansion of [h,g,g,g] we have the commutators of weight four. The product of these is

$$([h,x,x,y][h,x,y,x][h,y,x,x])^{a^2b} = 1.$$

Hence H is a right 3-Engel subgroup of G.

Finally we check that $[h,x,y,x,x] \neq 1$. Consider the normal closure of all the products of commutators said to be trivial in the definition of G. Any other trivial element of H would have to belong to this. Notice that of these products of commutators, and those obtained by commuting these by x, both $[h,x,y,x,x]$ and $[h,x,x,y,x]$ only occur together and with the same power. Hence neither can be trivial.

We have shown that Theorem 6.6 can't be improved by reducing the number of $\langle x\rangle^G$ entries. We end this section by showing that it was necessary to have that the subgroup is 3-torsion-free. Let $n \in \mathbb{N}$ be arbitrary. We give an example of a finite 3-group G with a normal right 3-Engel subgroup H and element x such that $[H,_n \langle x\rangle^G] \neq \{1\}$.

Let $F = \langle x_1, x_2, \ldots, x_n\rangle$ be the relatively free group of rank n and nilpotency class 2 and let $M = F/F^3$. Consider $G = C_3 \operatorname{wr} M = \prod_{g\in M} \langle a^g\rangle \rtimes M$. Letting $H = \prod_{g\in M} \langle a^g\rangle$, $h,k \in H$ and $g \in M$ we have that

$$[h,kg,kg,kg] = [h,g,g,g] = h^{(-1+g)^3} = h^{-1+g^3} = 1.$$

Hence H is a normal right 3-Engel subgroup of G. Note that G is finite and hence H is upper central. Also,

$$[a^1,[x_1,x_2],[x_1,x_3],\ldots,[x_1,x_n]] = a^{(-1+[x_1,x_2])\cdots(-1+[x_1,x_n])},$$

which is non-trivial as, for example, the $\langle a^1\rangle$ part is. As $[x_1,x_i] \in \langle x_1\rangle^G$, for each i, we have $[H,_n \langle x\rangle^G] \neq \{1\}$.

6.4 Bounding $\hat{c}_2(3)$ from above

In this section we use the results of the previous section to find an upper bound for $\hat{c}_2(3)$. We will show that $\hat{c}_2(3)$ is bounded above by 8. Whilst $\hat{c}_2(3)$ is a number associated with torsion-free subgroups, we only exclude the primes that are necessary to exclude. This gives us more information about right 3-Engel subgroups as discussed in Section 6.1. In order to use Theorem 6.6 we will need to exclude the prime 3. In fact this is necessary, as shown by the example at the end of the previous section. We will also need to exclude 2 and 5 as these are exceptional for 3-Engel groups. For now we exclude 2, as well as 3, which will give us more structure on $\langle h, x, y\rangle$. In particular we show with the following lemma that the right 3-Engel subgroup has upper central degree at most 6 in this case. Here we again impose that H is abelian, but as we will see later, this condition is easier to remove in this case. For a set X and positive integer i we will use the notation $[h,_i X]$ to denote the set of left normed commutators with first entry h and i entries from X.

Lemma 6.7. *Let H be an upper central normal right 3-Engel subgroup of a group G. Suppose that H is abelian and $\{2,3\}$-torsion-free. Let $x, y \in G$. Then $[H,_6 \langle x, y\rangle] = 1$.*

Proof. By Theorem 6.6 we know that $[H,_7 \langle x, y\rangle] = 1$. Let $h \in H$ and $g \in [h,_6 \{x, y\}]$, where g has three x and three y entries. It suffices to see that $g = 1$. Without loss of generality we may assume that the second entry is an x. Expanding $1 = [h, [x, y], [x, y], [x, y]]$ gives

$$\begin{aligned}1 =&[h, x, y, x, y, x, y][h, x, y, x, y, y, x]^{-1}[h, x, y, y, x, x, y]^{-1}[h, x, y, y, x, y, x]\\ &[h, y, x, x, y, x, y]^{-1}[h, y, x, x, y, y, x][h, y, x, y, x, x, y][h, y, x, y, x, y, x]^{-1}.\end{aligned}$$

By Lemma 6.2(iii) and repeated use of Lemma 6.3(iii) this becomes

$$1 =[h, x, y, x, y, y, x]^{-3}[h, x, y, y, x, y, x]^{3}.$$

Hence by Lemma 6.2(ii), $1 = [h, x, y, y, x, y, x]^6 = [h, x, y, y, x, y, x]$. By Lemma 6.3(iii) and Lemma 6.2(ii) we have

$$[h, x, y, y, x, x, y] = [h, x, y, x, y, y, x] = [h, x, y, y, x, y, x]^{-1} = 1.$$

By Lemma 6.2(ii), Lemma 6.3(iii) and Lemma 6.1(ii) respectively,

$$\begin{aligned}[h,x,x,y,x,y,y] =&[h,x,x,y,y,x,y]^{-1}\\ =&[h,x,y,x,x,y,y]^{-1}\\ =&[h,x,y,x,y,x,y][h,x,y,y,x,y,x]^{-1}\\ =&1.\end{aligned}$$

Thus g is trivial. □

We now introduce another element and consider $\langle h,x,y,z\rangle$. We show that commutators in $[h,_6\{x,y,z\}]$ with a triple entry are trivial, as long as $[H,_7\langle x,y,z\rangle]$ is. By Lemma 6.7 and Theorem 6.6, we need only consider commutators of multiweight (1,3,2,1) in h,x,y,z. We will then use this to show that indeed $[H,_7\langle x,y,z\rangle]=\{1\}$.

We first introduce some notation to make certain calculations easier to read. We write $(a_1;a_2;\ldots;a_n)$ for an arbitrary commutator starting with h and containing entries $a_1,\ldots,a_n$ in that order, with possibly other entries too. Within a calculation the remainder of the commutator is assumed to stay the same.

Lemma 6.8. *Let H be an upper central normal right 3-Engel subgroup of a group G. Suppose that H is abelian and $\{2,3\}$-torsion-free. Let $h\in H$ and $x,y,z\in G$. Suppose that $[H,_7\langle x,y,z\rangle]=\{1\}$. Then any commutator of multiweight (1,3,2,1) in h,x,y,z is trivial.*

Proof. Consider such a commutator with first entry h. Suppose that the second entry is $a\neq x$. By Theorem 6.6, $(ax;ax;ax;ax)=1$ and so, modulo commutators of multiweight (1,3,2,1) in h,x,y,z with first entry h and second entry x, $1=(a;x;x;x)uv$, where u consists of commutators with one x entry and v consists of commutators with two x entries. Replacing x by x^{-1} gives $1=(a;x;x;x)^{-1}u^{-1}v$. Hence, $1=(a;x;x;x)^2u^2$. Replacing x by x^2 now gives $1=(a;x;x;x)^{16}u^4$. Hence, $1=(a;x;x;x)^{12}$ and so, as H is {2,3}-torsion-free, we may assume that the second entry is an x.

Next consider the y and z entries. Replacing these by yz we have that $(yz;yz;yz)=1$, by Lemma 6.7. Hence

$$1=(y;y;z)(y;z;y)(z;y;y)(y;z;z)(z;y;z)(z;z;y).$$

Replacing y by y^{-1} and multiplying gives

$$1 = (y; y; z)^2 (y; z; y)^2 (z; y; y)^2 = (y; y; z)(y; z; y)(z; y; y).$$

So we may assume that the z entry is not before the first y entry.

Next we show that we may also assume that x is the third entry. So suppose not and that the last two x entries aren't consecutive, else by Lemma 6.2(ii) we could move one third. If the last two x's are within 3 entries, then we can move them together by Lemma 6.3(ii). Hence it remains to look at the commutators $[h, x, y, x, y, z, x]$ and $[h, x, y, x, z, y, x]$. But, by Lemma 6.3(ii),

$$[h, x, y, x, y, z, x] = [h, x, x, y, y, z, x]^{-1}[h, x, y, y, x, z, x]^{-1}$$

and by Lemma 6.3(iii) with y replaced by yz,

$$[h, x, y, x, z, y, x] = [h, x, y, x, y, z, x]^{-1}[h, x, y, y, x, x, z][h, x, y, z, x, x, y].$$

Hence we are left with the following commutators.

$$\begin{aligned} g_1 &= [h, x, x, y, x, y, z]. \\ g_2 &= [h, x, x, y, x, z, y]. \\ g_3 &= [h, x, x, y, y, x, z]. \\ g_4 &= [h, x, x, y, y, z, x]. \\ g_5 &= [h, x, x, y, z, x, y]. \\ g_6 &= [h, x, x, y, z, y, x]. \end{aligned}$$

We find relations involving these to show that they are trivial.

First notice that by Lemma 6.3(ii) $g_1 g_3 = 1$ (α). Next note that (6.5) gives $g_2 g_4 g_5 g_6 = 1$ (β). By Lemma 6.3(ii),

$$\begin{aligned} 1 =& [h, x, x, y, y, [x, z]][h, x, x, y, [x, z], y][h, x, x, [x, z], y, y] \\ =& g_3 g_4^{-1} g_2 g_5^{-1} [h, x, x, z, x, y, y]^{-1} \\ =& g_3 g_4^{-1} g_2 g_5^{-1} [h, x, x, y, x, z, y][h, x, x, y, x, y, z] \\ =& g_1 g_2^2 g_3 g_4^{-1} g_5^{-1} \ (\gamma). \end{aligned}$$

Also,

$$
\begin{aligned}
1 =&[h,x,x,y,y,x,z][h,x,x,y,y,z,x][h,x,z,y,y,x,x][h,z,x,y,y,x,x]\\
=&g_3g_4[h,x,x,z,y,y,x]^{-1}[h,z,x,x,y,y,x]^{-1}\\
=&g_3g_4[h,x,z,x,y,y,x]\\
=&g_3g_4[h,x,z,y,x,x,y]\\
=&g_3g_4[h,x,x,z,y,x,y]^{-1}\\
=&g_3^2g_4g_5 \ (\delta)
\end{aligned}
$$

and

$$
\begin{aligned}
1 =&[h,x,y,x,x,[y,z]][h,x,y,x,[y,z],x][h,x,y,[y,z],x,x]\\
=&[h,x,y,x,x,y,z][h,x,y,x,x,z,y]^{-1}[h,x,y,x,y,z,x][h,x,y,x,z,y,x]^{-1}\\
&[h,x,y,y,z,x,x][h,x,y,z,y,x,x]^{-1}\\
=&[h,x,x,y,x,y,z]^{-1}[h,x,x,y,x,z,y][h,x,y,x,y,z,x]^2[h,x,y,y,x,x,z]^{-1}\\
&[h,x,y,z,x,x,y]^{-1}[h,x,x,y,y,z,x]^{-1}[h,x,x,y,z,y,x]\\
=&g_1^{-1}g_2[h,x,y,y,x,z,x]^{-2}[h,x,x,y,y,z,x]^{-2}[h,x,x,y,y,x,z]\\
&[h,x,x,y,z,x,y]g_4^{-1}g_6\\
=&g_1^{-1}g_2[h,x,y,y,x,x,z]^2[h,x,y,y,z,x,x]^2g_4^{-2}g_3g_5g_4^{-1}g_6\\
=&g_1^{-1}g_2g_3^{-1}g_4^{-5}g_5g_6 \ (\epsilon).
\end{aligned}
$$

Now, $(\alpha)(\beta)^{-1}(\epsilon)$ implies $1 = g_4^{-6}$ and so $g_4 = 1$. By the same argument, reversing the order of the entries except h in all commutators, we get $1 = [h,x,z,y,y,x,x]$ and so

$$1 = [h,x,x,z,y,y,x]^{-1} = [h,x,x,y,z,y,x] = g_6.$$

Then $(\alpha)^{-1}(\beta)(\gamma)$ gives $1 = g_2^3 = g_2$ and (β) gives $g_5 = 1$. (δ) then gives $1 = g_3^2 = g_3$ and so (α) gives $g_1 = 1$, which completes the proof. □

We will need to remove the condition $[H,_7 \langle x,y,z\rangle] = \{1\}$ to use this lemma later on. We will now use what we have just proved to show that $[H,_7 \langle x,y,z\rangle] = \{1\}$.

Lemma 6.9. *Let H be an upper central normal right 3-Engel subgroup of a group G. Suppose that H is abelian and $\{2,3\}$-torsion-free. Let $x,y,z \in G$. Then $[H,_7 \langle x,y,z\rangle] = \{1\}$.*

Proof. We may assume that $[H,_8 \langle x,y,z\rangle] = \{1\}$. Let $h \in H$ and consider an arbitrary commutator in $[h,_7 \{x,y,z\}]$. By Lemma 6.8 we may assume that there is no triple entry within six consecutive entries. Thus we need only consider commutators of multiweight (1,3,2,2) in h,x,y,z with second and eighth entries both x.

Suppose that the third entry is y. Then by Lemma 6.8 we have

$$1 = [h, xy, xy, \ldots, xy, \ldots],$$

where the third xy is in place of the second x entry. Expanding this we have

$$1 = [h, x, x, \ldots, y, \ldots][h, x, y, \ldots, x, \ldots].$$

Hence we may assume that the third entry is also x. Assuming without loss of generality that the fourth entry is a y, this leaves the following commutators.

$$\begin{aligned} g_1 &= [h, x, x, y, y, z, z, x]. \\ g_2 &= [h, x, x, y, z, y, z, x]. \\ g_3 &= [h, x, x, y, z, z, y, x]. \end{aligned}$$

By Lemma 6.3(ii) on the last three entries, $g_1 = 1$. Then using Lemma 6.3(ii) on the fifth to seventh entries, $g_2 g_3 = 1$. Finally, by Lemma 6.3(ii),

$$1 = [h, x, x, y, z, z, [y, x]][h, x, x, y, z, [y, x], z][h, x, x, y, [y, x], z, z] = g_3.$$

Thus $g_2 = g_3^{-1} = 1$ and the proof is complete. □

In particular we have shown that commutators of multiweight (1,3,2,1) or higher in h,x,y,z are trivial. We now show that commutators in $[H,_7 G]$ with a triple entry are trivial as long as commutators of higher multiweight are.

Lemma 6.10. *Let H be an upper central normal right 3-Engel subgroup of a group G. Suppose that H is abelian and $\{2,3\}$-torsion-free. Then any commutator containing an element of H, a triple entry and four other entries is trivial if commutators of higher multiweight are.*

Proof. Consider commutators with first entry $h \in H$, a triple entry of $x \in G$ and other entries $a, b, c, d \in G$. Suppose that higher multiweight commutators are trivial. As in the proof of Lemma 6.8, we may assume that the second entry is an x. Also, by Lemma

6.3(ii),

$$\begin{aligned}
1 =&[h,x,a,x,x,b]^2[h,x,a,x,b,x]^2[h,x,a,b,x,x]^2[h,x,a,x,x,b,b]^{-1}\\
&[h,x,a,b,x,b,x]\\
=&[h,x,a,x,x,b]^2[h,x,a,x,b,x]^2[h,x,a,b,x,x]^2, \text{ by Lemma 6.8,}\\
=&[h,x,a,x,x,b][h,x,a,x,b,x][h,x,a,b,x,x]\\
=&[h,x,x,a,x,b]^{-1}[h,x,a,x,b,x][h,x,x,a,b,x]^{-1}.
\end{aligned}$$

Replacing a and b by arbitrary products we see that we may assume that the third entry is also x. So it suffices to see that the following commutators are trivial.

$$\begin{aligned}
g_1 &= [h,x,x,a,x,b,c,d]\\
g_2 &= [h,x,x,a,b,x,c,d]\\
g_3 &= [h,x,x,a,b,c,x,d]\\
g_4 &= [h,x,x,a,b,c,d,x].
\end{aligned}$$

Note that, by Lemma 6.8, g_1, g_2 and g_3 are alternating in the entries a, b and c. Further,

$$1 = [h,x,x,ad,x,[b,c],ad] = [h,x,x,ad,x,b,c,ad]^2,$$

$$1 = [h,x,x,[a,b],x,cd,cd] = [h,x,x,a,b,x,cd,cd]^2,$$

$$1 = [h,x,x,ad,[b,c],x,ad] = [h,x,x,ad,b,c,x,ad]^2.$$

Hence, as H is 2-torsion-free, g_1, g_2 and g_3 are alternating in the d entries as well. Now, again by Lemma 6.8,

$$1 =[h,x,x,a,x,[b,c],d][h,x,x,a,x,d,[b,c]] = g_1^4.$$

and hence $g_1 = 1$. Also,

$$\begin{aligned}
1 =&[h,x,x,[a,b],x,c,d][h,x,x,c,x,[a,b],d] = g_2^2,\\
1 =&[h,x,x,a,[b,c],x,d][h,x,x,[b,c],a,x,d] = g_3^4.
\end{aligned}$$

Thus $g_1 = g_2 = g_3 = 1$.

It remains to see that $g_4 = 1$. Now, by (6.5),

$$\begin{aligned} 1 =&[h,x,x,a,b,c,d,x][h,x,x,a,b,c,x,d][h,x,x,a,b,d,c,x] \\ &[h,x,x,a,b,d,x,c][h,x,x,a,b,x,c,d][h,x,x,a,b,x,d,c] \\ =&[h,x,x,a,b,c,d,x][h,x,x,a,b,d,c,x]. \end{aligned}$$

Thus g_4 is alternating in the c and d entries. Also,

$$\begin{aligned} 1 =&[h,x,x,a,b,c,[d,x]][h,x,x,a,b,[d,x],c][h,x,x,a,c,b,[d,x]] \\ &[h,x,x,a,c,[d,x],b][h,x,x,a,[d,x],b,c][h,x,x,a,[d,x],c,b] \\ =&[h,x,x,a,b,c,d,x][h,x,x,a,c,b,d,x]. \end{aligned}$$

Thus g_4 is also alternating in the b entry. Hence, by Lemma 6.8,

$$1 =[h,x,x,a,b,[c,d],x][h,x,x,a,[c,d],b,x] = g_4^4 = g_4.$$

This completes the proof. □

We now use this to show that the same is true in $[H,_8 G]$. Since we are aiming to show that under certain conditions $[H,_8 G] = \{1\}$, and we always assume H to be upper central, we can assume from now on that $[H,_9 G] = \{1\}$.

Corollary 6.11. *Let H be a normal right 3-Engel subgroup of a group G. Further, suppose that H is $\{2,3\}$-torsion-free, abelian and that $[H,_9 G] = \{1\}$. Then any commutator in $[H,_8 G]$ with a triple entry is trivial.*

Proof. Consider commutators with first entry $h \in H$, a triple entry of $x \in G$ and other entries $a,b,c,d,e \in G$. As in the proof of Lemma 6.10, we may assume that the second and third entry are both x. By Lemma 6.10 we are left with the commutator

$$g = [h,x,x,a,b,c,d,e,x].$$

First, by (6.5),

$$\begin{aligned} 1 =&[h,x,x,a,b,c,d,e,x][h,x,x,a,b,c,d,x,e][h,x,x,a,b,c,e,d,x] \\ &[h,x,x,a,b,c,e,x,d][h,x,x,a,b,c,x,d,e][h,x,x,a,b,c,x,e,d] \\ =&[h,x,x,a,b,c,d,e,x][h,x,x,a,b,c,e,d,x]. \end{aligned}$$

Thus g is alternating in the d and e entries. Also,

$$\begin{aligned}1 =&[h,x,x,a,b,c,d,[e,x]][h,x,x,a,b,c,[e,x],d][h,x,x,a,b,d,c,[e,x]]\\&[h,x,x,a,b,d,[e,x],c][h,x,x,a,b,[e,x],c,d][h,x,x,a,b,[e,x],d,c]\\=&[h,x,x,a,b,c,d,e,x][h,x,x,a,b,d,c,e,x]\end{aligned}$$

and thus g is also alternating in the c entry. Hence, again by (6.5), we have

$$\begin{aligned}1 =&[h,x,x,a,b,c,[d,e],x][h,x,x,a,b,c,x,[d,e]][h,x,x,a,b,[d,e],c,x]\\&[h,x,x,a,b,[d,e],x,c][h,x,x,a,b,x,c,[d,e]][h,x,x,a,b,x,[d,e],c]\\=&[h,x,x,a,b,c,[d,e],x][h,x,x,a,b,[d,e],c,x] = g^4 = g.\end{aligned}$$

□

We have seen that any commutator in $[H,_8 G]$ with a triple entry is trivial. The next step is to show that commutators with a double entry are trivial, which we prove in the following lemma. We impose that H is 5-torsion-free at this point.

Lemma 6.12. *Let H be a normal right 3-Engel subgroup of a group G. Further, suppose that H is $\{2,3,5\}$-torsion-free, abelian and that $[H,_9 G] = \{1\}$. Then any commutator in $[H,_8 G]$ with a double entry is trivial.*

Proof. Consider commutators with first entry $h \in H$, double entry x and other entries a, b, c, d, e and f. By Corollary 6.11 we may assume that the second entry is x, by use of $1 = (a;x;x)(x;a;x)(x;x;a)$. First we show that it suffices to see that

$$\begin{aligned}g_1 =&[h,x,x,a,b,c,d,e,f],\\g_2 =&[h,x,a,x,b,c,d,e,f],\\g_3 =&[h,x,a,b,x,c,d,e,f],\end{aligned}$$

are trivial. By Corollary 6.11,

$$(a;b;x)(a;x;b)(b;a;x)(b;x;a)(x;a;b)(x;b;a) = 1.$$

Hence, modulo commutators with second entry x and other x entry further to the left, the entries between the two x entries are alternating. So, modulo these commutators,

$$\begin{aligned}1 =& [\ldots,[a,b],c,x,\ldots][\ldots,[a,b],x,c,\ldots][\ldots,c,[a,b],x,\ldots] \\ & [\ldots,c,x,[a,b],\ldots][\ldots,x,[a,b],c,\ldots][\ldots,x,c,[a,b],\ldots] \\ =& [\ldots,a,b,c,x,\ldots]^2[\ldots,c,a,b,x,\ldots]^2 \\ =& [\ldots,a,b,c,x,\ldots]^4.\end{aligned}$$

Hence, as H is 2-torsion-free, it does indeed suffice to see that $g_1 = g_2 = g_3 = 1$.

Next suppose that g_1 and g_2 are trivial. Then g_3 is alternating in the entries a and b. Hence, by Lemma 6.3(ii),

$$\begin{aligned}1 =& [h,x,x,[a,b],c,d,e,f]^2[h,x,[a,b],x,c,d,e,f]^2[h,[a,b],x,x,c,d,e,f]^2 \\ =& g_3^4[h,a,b,x,x,c,d,e,f]^2[h,b,a,x,x,c,d,e,f]^{-2} \\ =& g_3^4[h,x,b,a,x,c,d,e,f]^{-2}[h,x,a,b,x,c,d,e,f]^2, \text{ by Corollary 6.11,} \\ =& g_3^8.\end{aligned}$$

Thus it remains to show that $g_1 = g_2 = 1$. To show this we can instead, by Lemma 6.3(ii), show that g_1 and $\hat{g}_2 = [h,a,x,x,b,c,d,e,f]$ are trivial. For this we will show that g_1 is alternating in the entries a,b,c,d,e,f and $\hat{g}_2$ in the entries b,c,d,e,f. By the proof that we need only check that g_1 and $\hat{g}_2$ are trivial, it suffices for this to show that the elements $[h,x,x,a,y,y,b,c,d]$, $[h,x,x,y,a,y,b,c,d]$, $[h,a,x,x,b,y,y,c,d]$ and $[h,a,x,x,y,b,y,c,d]$ are trivial.

Consider the following commutators in $[H,_8 G]$, where $k \in H$.

$$\begin{aligned}h_1 &= [k,x,x,a,y,y,\ldots], \\ h_2 &= [k,x,x,y,a,y,\ldots], \\ h_3 &= [k,x,a,x,y,y,\ldots], \\ h_4 &= [k,x,a,y,x,y,\ldots], \\ h_5 &= [k,x,y,x,a,y,\ldots], \\ h_6 &= [k,x,y,a,x,y,\ldots].\end{aligned}$$

The four commutators listed earlier are all of the same form as h_1 or h_2. Thus we would like to see that $h_1 = h_2 = 1$. First, by (6.5), $1 = h_1h_2h_3h_4h_5h_6$ (α). We also have

$$
\begin{aligned}
1 =&[k,x,x,[y,a,y],\ldots][k,x,[y,a,y],x,\ldots][k,[y,a,y],x,x,\ldots]\\
=&[k,x,x,y,a,y,\ldots]^3[k,x,y,a,y,x,\ldots]^3[k,y,a,y,x,x,\ldots]^3\\
=&h_2^3h_1^{-3}h_6^{-3}h_3^3\\
=&h_1^{-1}h_2h_3h_6^{-1}\ (\beta),
\end{aligned}
$$

$$
\begin{aligned}
1 =&[k,x,x,[a,y],y,\ldots][k,x,[a,y],x,y,\ldots][k,[a,y],x,x,y,\ldots]\\
=&h_1h_2^{-1}h_4h_6^{-1}[k,a,y,x,x,y,\ldots][k,y,a,x,x,y,\ldots]^{-1}\\
=&h_1h_2^{-1}h_3h_4^2h_5^{-1}h_6^{-2}\ (\gamma),
\end{aligned}
$$

$$
\begin{aligned}
1 =&[k,x,[x,a],y,y,\ldots][k,x,y,[x,a],y,\ldots][k,x,y,y,[x,a],\ldots]\\
=&h_1h_3^{-1}h_5h_6^{-1}[k,x,y,y,x,a,\ldots][k,x,y,y,a,x,\ldots]^{-1}\\
=&h_1h_2h_3^{-1}h_4^{-1}h_5^2h_6^{-2}\ (\delta),
\end{aligned}
$$

$$
\begin{aligned}
1 =&[k,x,x,y,[a,y],\ldots][k,x,[a,y],y,x,\ldots][k,[a,y],x,y,x,\ldots], \text{ by Lemma 6.10,}\\
=&h_2[k,x,x,y,y,a,\ldots]^{-1}[k,x,a,y,y,x,\ldots][k,x,y,a,y,x,\ldots]^{-1}\\
&[k,a,y,x,y,x,\ldots][k,y,a,x,y,x,\ldots]^{-1}\\
=&h_1^2h_2h_3^{-1}h_4^{-3}h_6\ (\epsilon).
\end{aligned}
$$

Now, $(\alpha)^5(\beta)^{-5}(\gamma)^7(\delta)(\epsilon)^6$ gives $h_1^{30} = 1$ and hence, as H is $\{2,3,5\}$-torsion-free, $h_1 = 1$. Then $(\alpha)(\beta)^{-1}(\gamma)(\epsilon)$ gives $h_6 = 1$. (β) is then $h_2h_3 = 1$, $(\alpha)(\beta)^{-1}$ is $h_4h_5 = 1$ and $(\alpha)(\gamma)$ is $h_3^2h_4^3 = 1$.

Now consider commutators in $[H,_8 G]$ with first entry h and double entries of x, y and z within six entries. We will show that these are trivial. First we may assume, by Corollary 6.11, that of these six entries, the first is x, fifth is y and sixth is z. Since $h_1 = h_6 = 1$ we are left with

$$
\begin{aligned}
k_1 &= [k,x,x,y,z,y,z,\ldots],\\
k_2 &= [k,x,y,x,z,y,z,\ldots],\\
k_3 &= [k,x,z,x,y,y,z,\ldots].
\end{aligned}
$$

By (6.5) we get $k_1k_2k_3 = 1$. Also, $h_2h_3 = 1$ gives $k_1k_3 = 1$. Hence $k_2 = 1$. Further, $h_3^2h_4^3 = 1$ implies that $k_3^2 = 1$ and hence that $k_3 = 1$. Thus $k_1 = 1$ as well. So these commutators are trivial. In particular, any commutator in $[H,_8 G]$ with first entry h

and double entries of x and y within six other entries will be alternating in the other two entries of the six.

By Lemma 6.10,

$$\begin{aligned}1 =& [h, x, y, y, x, [a, b], c, d][h, x, y, y, [a, b], x, c, d][h, [a, b], y, y, x, x, c, d] \\ =& [h, x, y, y, x, a, b, c, d]^2[h, x, y, y, a, b, x, c, d]^2[h, a, b, y, y, x, x, c, d]^2 \\ =& [h, x, y, y, x, a, b, c, d]^2[h, x, y, y, x, b, a, c, d]^{-2}[h, a, y, y, x, b, x, c, d]^{-2} \\ & [h, a, y, b, y, x, x, c, d]^{-2}, \text{ since } h_1 = 1, \\ =& [h, x, y, y, x, a, b, c, d]^4[h, a, y, y, x, b, x, c, d]^{-2}[h, a, y, b, y, x, x, c, d]^{-2}.\end{aligned}$$

Since $h_2h_3 = 1$, we get $[h, x, y, y, x, a, b, c, d]^4 = [h, x, y, y, x, a, b, c, d] = 1$. Hence, as $h_6 = 1$, $[h, x, a, y, x, y, b, c, d] = 1$. From $h_3^2h_4^3 = 1$ and $h_2h_3 = 1$ it follows that $h_2^{-2}h_4^3 = 1$. Hence $[h, x, x, y, a, y, b, c, d] = 1$ and similarly $[h, a, x, x, y, b, y, c, d] = 1$.

This proves that g_1 and $\hat{g}_2$ are alternating in the entries b, c, d, e, f. Thus, for g_1 and $\hat{g}_2$,

$$\begin{aligned}1 =& [h, \ldots, [b, c], [d, e], f][h, \ldots, [b, c], f, [d, e]][h, \ldots, [d, e], [b, c], f] \\ & [h, \ldots, [d, e], f, [b, c]][h, \ldots, f, [b, c], [d, e]][h, \ldots, f, [d, e], [b, c]] \\ =& [h, \ldots, b, c, d, e, f]^{24}\end{aligned}$$

Hence $g_1 = \hat{g}_2 = 1$, which completes the proof. □

We now have that commutators in $[H,_8 G]$ with a double entry are trivial. This shows that commutators in $[H,_8 G]$ are alternating in all entries, except for the entry from H. It will follow easily from this that $[H,_8 G] = \{1\}$, when H is abelian, as we now see.

Lemma 6.13. *Let G be a group and H be a normal right 3-Engel subgroup of G which is abelian, $\{2,3,5\}$-torsion-free and uppercentral. Then $[H,_8 G] = \{1\}$.*

Proof. We assume that $[H,_9 G] = \{1\}$. By the previous lemma, commutators of weight

9 and first entry in H are alternating in the other entries. Let $a_1, \dots, a_8 \in G$. Then,

$$\begin{aligned} 1 =& [h, a_1, [a_2, a_3], [a_4, a_5], a_6, a_7, a_8][h, a_1, [a_4, a_5], [a_2, a_3], a_6, a_7, a_8] \\ & [h, [a_2, a_3], a_1, [a_4, a_5], a_6, a_7, a_8][h, [a_2, a_3], [a_4, a_5], a_1, a_6, a_7, a_8] \\ & [h, [a_4, a_5], a_1, [a_2, a_3], a_6, a_7, a_8][h, [a_4, a_5], [a_2, a_3], a_1, a_6, a_7, a_8] \\ =& [h, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]^{24}. \end{aligned}$$

As H is {2,3}-torsion-free we are done. □

We now remove the condition H is abelian, to give our main result of the section, which shows that $\hat{c}_2(3) \leq 8$.

Theorem 6.14. *Let G be a group and H be a normal right 3-Engel subgroup of G which is {2,3,5}-torsion-free and uppercentral. Then $[H,_8 G] = \{1\}$.*

Proof. Let $h \in H$, $g_1, \dots, g_8 \in G$ and $g = [h, g_1, \dots, g_8]$. Consider $K = G/H'$. The $\{2, 3, 5\}$-elements of H/H' form a normal subgroup, as H/H' is abelian. Quotienting K out by this subgroup we can apply Lemma 6.13 to get $g^l u = 1$, where $u \in H'$ and $l = 2^a 3^b 5^c$ for some $a, b, c \in \mathbb{N}$. We can expand u to write it as a product of commutators with at least two entries of h and at least one entry of each $g_1, \dots, g_8$. Hence $g^l = 1$ modulo higher multiweights. As H is upper central and {2,3,5}-torsion-free the result follows. □

6.5 Bounding $\hat{c}_2(3)$ from below

We now show that $\hat{c}_2(3) = 8$, by finding a group G and subgroup H satisfying the conditions of Theorem 6.14, but with $[H,_7 G] \neq \{1\}$. Let $G = \langle h, x, y, a, b, c \rangle$ be the free group on six generators. Let $H = \langle h \rangle^G$. Quotient G out by H' to make H abelian. Further, quotient out by any commutator containing a triple entry of x or y, or a double entry of either a, b or c. In particular it follows that $[H,_8 G] = \{1\}$ and H is uppercentral. Note that the discussion at the end of Section 2.3 shows that there are no relations between left normed commutators in H with first entry h and entry set $\{h, x, y, a, b, c\}$, other than those that follow directly from what is quotiented out by above. If we were to quotient out by the normal closure of every $[h, g, g, g]$, for $g \in G$, then H would be a right 3-Engel subgroup and we would need only check that there is a torsion-free element in $[H,_7 G]$. However, we do not yet quotient out by these normal closures, but instead place some more structure on $[H,_7 G]$ first.

Note that currently $[H,_7 G]$ is generated by left normed commutators with first entry h, double entries of both x and y and single entries of a, b and c, and that these commutators are independent. We quotient out by products of commutators to make all commutators in $[H,_7 G]$ alternating in entries of a, b and c and such that we can swap both x entries with both y entries. Further, we quotient out by every product of the form

$$[h, g_1, g_2, \dots, g_7][h, g_2, g_3, \dots, g_7, g_1]^{-1},$$

where $g_1, \dots, g_7 \in \{x, y, a, b, c\}$, so that the entries after h in commutators in $[H,_7 G]$ can be cycled around. We also quotient out by products of the form

$$[h, g_1, g_2, \dots, g_7][h, g_7, g_6, \dots, g_1],$$

so that the entries after h in commutators in $[H,_7 G]$ can be reflected, which inverses the commutator.

Using the alternating, cycling, reflecting and swapping properties we can seperate the generators of $[H,_7 G]$ into equivalence classes, where two commutators are equivalent if one is equal to the other or the inverse of the other. For each equivalence class we can pick as a representative the smallest commutator in the class with respect to the lexicographical order where $x < y < a < b < c$. Note that these representatives are independent. We now place these in sets as follows. Let

$$\begin{aligned}
A_{-4} &= \{[h, x, x, a, y, b, y, c]\} \\
A_{-3} &= \{[h, x, x, y, y, a, b, c]\} \\
A_{-2} &= \{[h, x, y, x, a, y, b, c], [h, x, y, a, y, x, b, c]\} \\
A_0 &= \{[h, x, y, a, x, y, b, c]\} \\
A_1 &= \{[h, x, x, y, a, y, b, c], [h, x, x, y, a, b, y, c], [h, x, y, x, y, a, b, c], [h, x, y, a, x, b, y, c]\} \\
A_2 &= \{[h, x, x, y, a, b, c, y], [h, x, x, a, y, y, b, c]\} \\
A_3 &= \{[h, x, y, a, y, b, x, c]\}.
\end{aligned}$$

For each $x_i \in A_i$ we quotient G out by $[h, x, x, y, a, y, b, c]^{-i} x_i$. Thus we have that $x_i = [h, x, x, y, a, y, b, c]^i$. Since the elements above were previously independent we have that $[h, x, x, y, a, y, b, c]$ is still a torsion-free element of $[H,_7 G]$.

Using the alternating, cycling, reflecting and swapping properties above, every commutator in $[H,_7 G]$ may be written as a product of elements from the A_i's. Thus $[H,_7 G]$

is generated by $[h, x, x, y, a, y, b, c]$. Let

$$N = \langle [h, g, g, g] : g \in G \rangle^G.$$

Then H/N is a right 3-Engel subgroup of G/N and it remains to check that no non-zero power of $[h, x, x, y, a, y, b, c]$ is in N. N is generated by elements $[h, g, g, g, k_1, \ldots, k_l]$, where $0 \leq l \leq 4$ and $k_1, \ldots, k_l \in \{x, y, a, b, c\}$, $g \in G$. We can expand these commutators so that they are products of left normed commutators with entries from $\{h, x, y, a, b, c\}$. Consider such a product. This product with any number of elements from $\{x, y, a, b, c\}$ replaced by 1 is either trivial or one of the generators. Thus we can choose the generators of N such that they are products of commutators with the same entry set.

Let $z = [h, x, x, y, a, y, b, c]$ and suppose, for a contradiction, that $z^j \in N$, for some $j \neq 0$. Then z^j can be written as a product of the generators discussed above, where the generators are products of commutators with at least one entry of each element from $I = \{h, x, y, a, b, c\}$. Suppose that

$$z^j = \alpha_1\beta_1\gamma_1\delta_1 \cdots \alpha_m\beta_m\gamma_m\delta_m,$$

where, for each $i \in \{1, \ldots, m\}$, $\alpha_i\beta_i\gamma_i\delta_i$ is such a generator or its inverse, α_i is a product of commutators with one entry of each element of I, β_i is a product of commutators with a double entry of x and one entry of each other element of I, γ_i is a product of commutators with a double entry of y and one entry of each other element of I and δ_i is a product of commutators with a double entry of x, a double entry of y and one entry of each other element of I.

For $g \in \{x, y\}$ we define a function D_g on products of left normed commutators with first entry h, where D_g replaces each entry of g in each commutator by two entries of g, e.g. $D_g([h, \ldots, g, \ldots]) = [h, \ldots, g, g, \ldots]$, where the entries of the commutator which aren't g are unchanged. Note that D_g is multiplicative and well-defined on products of left normed commutators with first entry h that are not in $[H,_7 G]$, as we have not added any relations here except that commutators with two h, a, b or c entries or three x or y entries are trivial. In fact it is also well defined on $[H,_7 G]$, as it maps all such commutators to the trivial element. Now, $\alpha_1 \cdots \alpha_m = 1$, $\beta_1 \cdots \beta_m = 1$, $\gamma_1 \cdots \gamma_m = 1$ and $\delta_1 \cdots \delta_m = z^j$. Thus

$$z^{4j} = D_x(D_y(\alpha_1 \cdots \alpha_m))(D_y(\beta_1 \cdots \beta_m)D_x(\gamma_1 \cdots \gamma_m))^2(\delta_1 \cdots \delta_m)^4$$

and hence

$$z^{4j} = D_x(D_y(\alpha_1))D_y(\beta_1)^2 D_x(\gamma_1)^2 \delta_1^4 \cdots D_x(D_y(\alpha_m))D_y(\beta_m)^2 D_x(\gamma_m)^2 \delta_m^4.$$

Thus at least one $D_x(D_y(\alpha_i))D_y(\beta_i)^2 D_x(\gamma_i)^2 \delta_i^4$ is non-trivial. We will show that this is not the case, which will give us the desired contradiction. So we need to check that the identity $D_x(D_y(\alpha_i))D_y(\beta_i)^2 D_x(\gamma_i)^2 \delta_i^4 = 1$ holds for every $\alpha_i\beta_i\gamma_i\delta_i$ as defined above. The calculations can be found in Appendix A.

Chapter 7

Further Work

Here we discuss some further directions in which one may expand upon the work in this thesis. One natural next step would be to find more values for the functions arising from the theorems in Chapters 3 and 4. The values for 2-Engel groups and right 2-Engel subgroups are all displayed in this thesis. For 3-Engel groups we have discussed all the values for functions arising from Theorems 3.1 - 3.4. Further work could be carried out to find exact values for the theorems of Section 3.5 on n-Engel p-groups when $n = 3$. For right 3-Engel subgroups many of the values remain unknown and could be investigated. A fair amount is known about 4-Engel groups and these could be investigated further to find the specific values for the functions arising from Chapter 3. For right 4-Engel subgroups one could try a similar approach to that taken for right 3-Engel subgroups in Chapter 6. However, this is likely to be a lot more complicated and so perhaps a different approach is necessary. Some information is known about the structure of 5-Engel groups and further study of these groups could reveal the values of the functions arising from Chapter 3. Little is known about n-Engel groups for $n > 5$.

A different direction to take this work would be to consider different theorems about n-Engel groups and see whether these can be generalised to results about right n-Engel subgroups. For example, a theorem of Medvedev [28] states that every compact n-Engel group is locally nilpotent and a theorem of Kim and Rhemtulla [26] states that every orderable n-Engel group is nilpotent. One could investigate whether these theorems can be generalised to apply to right n-Engel subgroups.

Appendix A

Calculations for Section 6.5

A.1 Introduction

Here we perform the calculations discussed in Section 6.5. We fully expand a general commutator of the form $[h, g, g, g, k_1, \ldots, k_l]$, where $0 \leq l \leq 4$ and $k_1, \ldots, k_l \in \{x, y, a, b, c\}$, $g \in G$. Then we set α to be the product of commutators in this expansion with one entry of each element of $I = \{h, x, y, a, b, c\}$, β to be the product of commutators with a double entry of x and one entry of each other element of I, γ to be the product of commutators with a double entry of y and one entry of each other element of I and δ to be the product of commutators with a double entry of x, a double entry of y and one entry of each other element of I. We then check that the identity

$$D_x(D_y(\alpha))D_y(\beta)^2 D_x(\gamma)^2 \delta^4 = 1 \tag{A.1}$$

holds.

A.2 Shortening the calculations

Before we start the calculations we can use some properties of the group to shorten the workload. To make use of the reflecting property of $[H,_7 G]$ the following Lemma will be helpful.

Lemma A.1. *Let $h, a_1, \ldots, a_n$ be elements of a group, for $n \geq 2$. Then, modulo higher multiweights, $[h, [a_1, \ldots, a_n]]$ can be written in the form*
(i) $[h, g_{1,1}, \ldots, g_{1,n}][h, g_{1,n}, \ldots, g_{1,1}] \cdots [h, g_{r,1}, \ldots, g_{r,n}][h, g_{r,n}, \ldots, g_{r,1}]$ for n odd, and
(ii) $[h, g_{1,1}, \ldots, g_{1,n}][h, g_{1,n}, \ldots, g_{1,1}]^{-1} \cdots [h, g_{r,1}, \ldots, g_{r,n}][h, g_{r,n}, \ldots, g_{r,1}]^{-1}$ for n even.

Proof. We use induction on n. First, $[h, [a_1, a_2]] = [h, a_1, a_2][h, a_2, a_1]^{-1}$ modulo higher multiweights and so the claim is true for $n = 2$. Now suppose that the claim is true for $n = k - 1$, where $k > 2$. If k is odd, then

$$\begin{aligned}
[h, [a_1, \ldots, a_k]] =& [h, [a_1, \ldots, a_{k-1}], a_k][h, a_k, [a_1, \ldots, a_{k-1}]]^{-1} \\
=& [h, g_{1,1}, \ldots, g_{1,k-1}, a_k][h, g_{1,k-1}, \ldots, g_{1,1}, a_k]^{-1} \\
& \cdots [h, g_{r,1}, \ldots, g_{r,k-1}, a_k][h, g_{r,k-1}, \ldots, g_{r,1}, a_k]^{-1} \\
& [h, a_k, g_{1,1}, \ldots, g_{1,k-1}]^{-1}[h, a_k, g_{1,k-1}, \ldots, g_{1,1}] \\
& \cdots [h, a_k, g_{r,1}, \ldots, g_{r,k-1}]^{-1}[h, a_k, g_{r,k-1}, \ldots, g_{r,1}] \\
=& [h, g_{1,1}, \ldots, g_{1,k-1}, a_k][h, a_k, g_{1,k-1}, \ldots, g_{1,1}] \\
& \cdots [h, g_{r,1}, \ldots, g_{r,k-1}, a_k][h, a_k, g_{r,k-1}, \ldots, g_{r,1}] \\
& [h, g_{1,k-1}, \ldots, g_{1,1}, a_k]^{-1}[h, a_k, g_{1,1}, \ldots, g_{1,k-1}]^{-1} \\
& \cdots [h, g_{r,k-1}, \ldots, g_{r,1}, a_k]^{-1}[h, a_k, g_{r,1}, \ldots, g_{r,k-1}]^{-1} \\
=& [h, g_{1,1}, \ldots, g_{1,k-1}, a_k][h, a_k, g_{1,k-1}, \ldots, g_{1,1}] \\
& \cdots [h, g_{r,1}, \ldots, g_{r,k-1}, a_k][h, a_k, g_{r,k-1}, \ldots, g_{r,1}] \\
& [h, g_{1,k-1}, \ldots, g_{1,1}, a_k^{-1}][h, a_k^{-1}, g_{1,1}, \ldots, g_{1,k-1}] \\
& \cdots [h, g_{r,k-1}, \ldots, g_{r,1}, a_k^{-1}][h, a_k^{-1}, g_{r,1}, \ldots, g_{r,k-1}].
\end{aligned}$$

If k is even, then

$$\begin{aligned}
[h, [a_1, \ldots, a_k]] =& [h, [a_1, \ldots, a_{k-1}], a_k][h, a_k, [a_1, \ldots, a_{k-1}]]^{-1} \\
=& [h, g_{1,1}, \ldots, g_{1,k-1}, a_k][h, g_{1,k-1}, \ldots, g_{1,1}, a_k] \\
& \cdots [h, g_{r,1}, \ldots, g_{r,k-1}, a_k][h, g_{r,k-1}, \ldots, g_{r,1}, a_k] \\
& [h, a_k, g_{1,1}, \ldots, g_{1,k-1}]^{-1}[h, a_k, g_{1,k-1}, \ldots, g_{1,1}]^{-1} \\
& \cdots [h, a_k, g_{r,1}, \ldots, g_{r,k-1}]^{-1}[h, a_k, g_{r,k-1}, \ldots, g_{r,1}]^{-1} \\
=& [h, g_{1,1}, \ldots, g_{1,k-1}, a_k][h, a_k, g_{1,k-1}, \ldots, g_{1,1}]^{-1} \\
& \cdots [h, g_{r,1}, \ldots, g_{r,k-1}, a_k][h, a_k, g_{r,k-1}, \ldots, g_{r,1}]^{-1} \\
& [h, g_{1,k-1}, \ldots, g_{1,1}, a_k]^{-1}[h, a_k, g_{1,1}, \ldots, g_{1,k-1}] \\
& \cdots [h, g_{r,k-1}, \ldots, g_{r,1}, a_k]^{-1}[h, a_k, g_{r,1}, \ldots, g_{r,k-1}].
\end{aligned}$$

□

Now consider the expansion of $[h, g, g, g, k_1, \ldots, k_l]$ for a general $g \in G$ and some $k_1, \ldots, k_l \in \{x, y, a, b, c\}$. Write g as a product of generators x, y, a, b and c and their inverses, say $g = g_1 \cdots g_m$. One needs only to consider the commutators in the expansion of $[h, g, g, g, k_1, \ldots, k_l]$ which contain entries of each $g_1, \ldots, g_m$, as expansions of the same form with g a shorter product will be considered seperately. Recall that commutators with a triple entry of x or y, or a double entry of either a, b or c is trivial. Thus we can assume that g is a product of at most two of each $\{x, x^{-1}\}$ and $\{y, y^{-1}\}$ and at most one of each a, b and c. Further, we can assume that the first entry in the product from $\{a, b, c\}$ is a, second b and third c, if in the product. If there are two elements from $\{x, x^{-1}\}$ in the product, then we can assume they are both x, by taking any inverses outside the commutators. Similarly for y.

Now suppose there is a single x^{-1} in the product and no x, but that one of $k_1, \ldots, k_l$ is x. Then the inverse can be taken outside the commutator and we can replace x^{-1} by x. Suppose that instead we have a single x in the product for g and no x^{-1} and $k_1, \ldots, k_l$ all not x. Suppose that (A.1) holds for this. Thus $D_x(D_y(\alpha))D_y(\beta)D_x(\gamma)\delta = 1$. Now, replacing x by x^{-1} changes $\alpha\beta\gamma\delta$ to $\alpha^{-1}D_x(\alpha)\beta\gamma^{-1}D_x(\gamma)\delta$. Then, checking $(A.1)$,

$$D_x(D_y(\alpha^{-1}))D_y(D_x(\alpha)\beta)^2 D_x(\gamma^{-1})^2 (D_x(\gamma)\delta)^4 = D_x(D_y(\alpha))D_y(\beta)^2 D_x(\gamma)^2\delta^4 = 1.$$

Thus we may assume that the product g contains no inverses. Suppose instead that we replace the x by x^2. Then $\alpha\beta\gamma\delta$ becomes $\alpha^2 D_x(\alpha)\beta^4\gamma^2 D_x(\gamma)\delta^4$ and, again checking (A.1),

$$D_x(D_y(\alpha^2))D_y(D_x(\alpha)\beta^4)^2 D_x(\gamma^2)^2 (D_x(\gamma)\delta^4)^4 = D_x(D_y(\alpha))^4 D_y(\beta)^8 D_x(\gamma)^8\delta^{16} = 1.$$

Thus we may assume that there is no occurence of x^2 or y^2 in the product for g. Further, if we allow left normed commutators with entries from $\{x, y, a, b, c\}$ in the product for g and we have a left normed commutator with an entry of x in a product for g, and no other x in the product, and (A.1) held, then replacing this x with x^2 shows that D_x of this product would also satisfy (A.1). Thus we needn't check products containing a left normed commutator with two consecutive x entries or y entries.

Suppose that if there is an x and a y in the product, then the x appears first. By Hall's collection process and the above we can write $g = tuvw$, where

$$\begin{aligned}
t \in &\{1, x, y, xy, [x, y], x[x, y], y[x, y], xy[x, y]\} \\
u \in &\{1, a, [a, x], [a, y], [a, x, y], [a, y, x], [a, x, y, x], [a, y, x, y], [a, x, y, x, y]\} \\
v \in &\{1, b, [b, x], [b, y], [b, x, y], [b, y, x], [b, x, y, x], [b, y, x, y], [b, x, y, x, y]\} \\
w \in &\{1, c, [c, x], [c, y], [c, x, y], [c, y, x], [c, x, y, x], [c, y, x, y], [c, x, y, x, y]\},
\end{aligned}$$

and $tuvw$ written out contains at most two x's and two y's. Note that the case $t = y$ follows from the case $t = x$ by swapping x and y. Similarly swapping x and y in the case $t = x[x,y]$ gives $t = y[y,x] = y[x,y]^{-1}$. Expanding with this t we only consider commutators with an entry of both $[x,y]^{-1}$ and y. So the inverse can be taken outside the commutators and we have the case $y[x,y]$. Thus we needn't check the cases $t = y$ and $t = y[x,y]$.

We will say that an element satisfies (A.1) if every element in its normal closure does. Now,

$$
\begin{aligned}
& [h, tuvw, tuvw, tuvw] \text{ satisfies (A.1) } \forall t,u,v,w \\
\iff & [h, tuv, tuv, tuv] \text{ and} \\
& [h, tuv, tuv, w][h, tuv, w, tuv][h, w, tuv, tuv] \\
& [h, tuv, tuv, tuv, w][h, tuv, tuv, w, tuv][h, tuv, w, tuv, tuv] \text{ satisfy (A.1) } \forall t,u,v,w \\
\iff & [h, tuv, tuv, tuv] \text{ and} \\
& [h, tuv, tuv, w][h, tuv, w, tuv][h, w, tuv, tuv] \text{ satisfy (A.1) } \forall t,u,v,w \\
\iff & [h, tu, tu, tu] \text{ and} \\
& [h, tu, tu, v][h, tu, v, tu][h, v, tu, tu] \text{ and} \\
& [h, tuv, tuv, w][h, tuv, w, tuv][h, w, tuv, tuv] \text{ satisfy (A.1) } \forall t,u,v,w \\
\iff & [h, t, t, t] \text{ and} \\
& [h, t, t, u][h, t, u, t][h, u, t, t] \text{ and} \\
& [h, tu, tu, v][h, tu, v, tu][h, v, tu, tu] \text{ and} \\
& [h, tuv, tuv, w][h, tuv, w, tuv][h, w, tuv, tuv] \text{ satisfy (A.1) } \forall t,u,v,w \\
\iff & [h, t, t, t] \text{ and} \\
& [h, t, t, u][h, t, u, t][h, u, t, t] \text{ and} \\
& [h,t,u,v][h,u,t,v][h,t,v,u][h,u,v,t][h,v,t,u][h,v,u,t][h,t,u,t,v][h,t,t,u,v] \\
& [h,t,u,v,t][h,t,v,t,u][h,v,t,u,t][h,v,t,t,u] \text{ and} \\
& [h,tu,v,w][h,v,tu,w][h,tu,w,v][h,v,w,tu][h,w,tu,v][h,w,v,tu][h,tu,v,tu,w] \\
& [h,tu,tu,v,w][h,tu,v,w,tu][h,tu,w,tu,v][h,w,tu,v,tu][h,w,tu,tu,v] \\
& \text{satisfy (A.1) } \forall t,u,v,w.
\end{aligned}
$$

Let (i) be the statement $[h,t,t,t]$ satisfies (A.1), for all t, (ii) be the statement $[h,t,t,u][h,t,u,t][h,u,t,t]$ satisfies (A.1), for all t,u and (iii) be the statement

$$
\begin{gathered}
[h,t,u,v][h,u,t,v][h,t,v,u][h,u,v,t][h,v,t,u][h,v,u,t] \\
[h,u,t,t,v]^{-1}[h,t,u,v,t][h,t,v,t,u][h,v,u,t,t]^{-1}
\end{gathered}
$$

satisfies (A.1), for all t, u, v. Then we have

$$
\begin{aligned}
& [h, tuvw, tuvw, tuvw] \text{ satisfies (A.1) } \forall t, u, v, w \\
\Longleftrightarrow\ & \text{(i),(ii) and (iii) hold and} \\
& [h, tu, v, w][h, v, tu, w][h, tu, w, v][h, v, w, tu][h, w, tu, v][h, w, v, tu][h, tu, v, tu, w] \\
& [h, tu, tu, v, w][h, tu, v, w, tu][h, tu, w, tu, v][h, w, tu, v, tu][h, w, tu, tu, v] \\
& \text{satisfies (A.1) } \forall t, u, v, w \\
\Longleftrightarrow\ & \text{(i),(ii) and (iii) hold and} \\
& [h, u, v, w][h, v, u, w][h, u, w, v][h, v, w, u][h, w, u, v][h, w, v, u] \text{ and} \\
& [h, t, u, v, w][h, v, t, u, w][h, t, u, w, v][h, v, w, t, u][h, w, t, u, v][h, w, v, t, u] \\
& [h, t, v, u, w][h, u, v, t, w][h, t, u, v, w][h, u, t, v, w][h, t, v, w, u][h, u, v, w, t] \\
& [h, t, w, u, v][h, u, w, t, v][h, w, t, v, u][h, w, u, v, t][h, w, t, u, v][h, w, u, t, v] \\
& [h, t, u, v, t, w][h, t, v, t, u, w][h, t, t, u, v, w][h, t, u, t, v, w][h, t, v, w, t, u] \\
& [h, t, u, v, w, t][h, t, w, t, u, v][h, t, u, w, t, v][h, w, t, v, t, u][h, w, t, u, v, t] \\
& [h, w, t, t, u, v][h, w, t, u, t, v] \text{ satisfy (A.1) } \forall t, u, v, w \\
\Longleftrightarrow\ & \text{(i),(ii) and (iii) hold and} \\
& [h, u, v, w][h, v, u, w][h, u, w, v][h, v, w, u][h, w, u, v][h, w, v, u] \text{ and} \\
& [h, t, u, v, w][h, v, t, u, w][h, v, w, t, u][h, w, t, u, v][h, w, v, t, u][h, u, v, t, w] \\
& [h, u, t, v, w][h, u, v, w, t][h, u, w, t, v][h, w, t, v, u][h, w, u, v, t][h, w, t, u, v] \\
& [h, w, u, t, v][h, t, w, v, u]^{-1}[h, t, u, v, t, w][h, t, v, t, u, w][h, u, t, t, v, w]^{-1} \\
& [h, t, v, w, t, u][h, t, u, v, w, t][h, t, w, t, u, v][h, t, u, w, t, v][h, w, t, v, t, u] \\
& [h, w, t, u, v, t][h, w, u, t, t, v]^{-1} \text{ satisfy (A.1) } \forall t, u, v, w.
\end{aligned}
$$

Using (iii) commuted by w and then also cycled so that the w is after the h, the last of these becomes

$$
\begin{aligned}
& [h, v, w, t, u][h, w, t, u, v][h, u, v, w, t][h, u, w, t, v][h, t, w, v, u]^{-1}[h, t, v, u, w]^{-1} \\
& [h, v, u, t, w]^{-1}[h, w, v, u, t]^{-1}[h, t, v, w, t, u][h, t, u, v, w, t][h, t, w, t, u, v][h, t, u, w, t, v] \\
& [h, v, u, t, t, w][h, w, v, u, t, t].
\end{aligned}
$$

Let (v) be the statement that this satisfies (A.1), for all t, u, v, w and (iv) the statement that

$$[h, u, v, w][h, v, u, w][h, u, w, v][h, v, w, u][h, w, u, v][h, w, v, u]$$

satisfies (A.1), for all u, v, w. It then remains to check that (i) - (v) hold.

A.3 Expanding commutators within commutators

Before starting the calculations for (i) - (v) we expand some commutators which will occur frequently. First, in G,

$$\begin{aligned}[h,[x,y]] =&[h,x,y][h,y,x]^{-1}[h,x,x,y]^{-1}[h,x,y,x][h,y,x,y]^{-1}[h,y,y,x]\\ &[h,x,y,y,x]^{-1}[h,x,y,x,y].\end{aligned}$$

Hence when checking (i) - (v), if we have another x and y in a commutator, we may replace $[\ldots,[x,y],\ldots]$ by $[\ldots,x,y,\ldots][\ldots,y,x,\ldots]^{-1}$. From the sets A_i defined in Section 6.5 one can check that, in $[H,_7 G]$,

$$[h,x,x,y,\ldots][h,x,y,x,\ldots][h,y,x,x,\ldots]=1.$$

This will also be checked in the proof of (i). Hence, if there is another x, but no other y, then we may replace $[\ldots,[x,y],\ldots]$ by

$$[\ldots,x,y,y,\ldots][\ldots,y,y,x,\ldots]^{-1}[\ldots,y,x,y,\ldots]^{-2}[\ldots,y,y,x,\ldots]^{2}=[\ldots,y,x,y,\ldots]^{-3}.$$

If there is another y, but no other x, then we may replace $[\ldots,[x,y],\ldots]$ by

$$[\ldots,x,x,y,\ldots][\ldots,y,x,x,\ldots]^{-1}[\ldots,x,y,x,\ldots]^{2}[\ldots,x,x,y,\ldots]^{-2}=[\ldots,x,y,x,\ldots]^{3}.$$

If there is no other x or y, then we may replace $[\ldots,[x,y],\ldots]$ by

$$\begin{aligned}&[\ldots,x,x,y,y,\ldots][\ldots,y,y,x,x,\ldots]^{-1}[\ldots,x,x,y,y,\ldots]^{-2}[\ldots,x,y,y,x,\ldots]^{2}\\ &[\ldots,y,x,x,y,\ldots]^{-2}[\ldots,y,y,x,x,\ldots]^{2}[\ldots,x,y,y,x,\ldots]^{-4}[\ldots,x,y,x,y,\ldots]^{4}\\ =&[\ldots,x,y,y,x,\ldots]^{-4}[\ldots,x,y,x,y,\ldots]^{4}.\end{aligned}$$

Next,

$$[h,[a,x]]=[h,a,x][h,x,a]^{-1}[h,x,x,a][h,x,a,x]^{-1}.$$

So if there is another x in a commutator, then we can replace $[\ldots,[a,x],\ldots]$ by $[\ldots,a,x,\ldots][\ldots,x,a,\ldots]^{-1}$. Again from the sets A_i one can check that

$$[h,x,x,a,\ldots][h,x,a,x,\ldots][h,a,x,x,\ldots]=1.$$

This will be checked in the proof of (ii). Hence, if there is no other x, then we may replace $[\ldots,[a,x],\ldots]$ by $[\ldots,x,a,x,\ldots]^{-3}$.

Also,

$$\begin{aligned}
[h,[a,x,y]] =&[h,[a,x],y][h,y,[a,x]]^{-1}[h,y,y,[a,x]][h,y,[a,x],y]^{-1}\\
=&[h,a,x,y][h,x,a,y]^{-1}[h,y,a,x]^{-1}[h,y,x,a][h,x,x,a,y][h,x,a,x,y]^{-1}\\
&[h,y,x,x,a]^{-1}[h,y,x,a,x][h,y,y,a,x][h,y,y,x,a]^{-1}[h,y,a,x,y]^{-1}\\
&[h,y,x,a,y][h,y,y,x,x,a][h,y,y,x,a,x]^{-1}[h,y,x,x,a,y]^{-1}[h,y,x,a,x,y]
\end{aligned}$$

Hence, if there is another x and y in a commutator, we can replace $[\ldots,[a,x,y],\ldots]$ by

$$[\ldots,a,x,y\ldots][\ldots,y,x,a,\ldots][\ldots,x,a,y\ldots]^{-1}[\ldots,y,a,x,\ldots]^{-1}.$$

If there is another x in a commutator, then we can replace $[\ldots,[a,x,y],\ldots]$ by

$$\begin{aligned}
&[\ldots,a,x,y,y,\ldots][\ldots,y,y,x,a,\ldots][\ldots,x,a,y,y,\ldots]^{-1}[\ldots,y,y,a,x,\ldots]^{-1}\\
&[\ldots,y,y,a,x,\ldots]^{2}[\ldots,y,y,x,a,\ldots]^{-2}[\ldots,y,a,x,y,\ldots]^{-2}[\ldots,y,x,a,y,\ldots]^{2}\\
=&[\ldots,a,x,y,y,\ldots][\ldots,x,a,y,y,\ldots]^{-1}[\ldots,y,y,a,x,\ldots][\ldots,y,y,x,a,\ldots]^{-1}\\
&[\ldots,y,a,x,y,\ldots]^{-2}[\ldots,y,x,a,y,\ldots]^{2}.
\end{aligned}$$

In the proof of (i) we will in particular prove that

$$[h,x,x,[a,y],\ldots][h,x,[a,y],x,\ldots][h,[a,y],x,x,\ldots]=1$$

and so, swapping the roles of x and y, we can replace $[\ldots,[a,x,y],\ldots]$ by

$$[\ldots,y,x,a,y,\ldots]^{3}[\ldots,y,a,x,y,\ldots]^{-3}.$$

If there is another y, then we can replace $[\ldots,[a,x,y],\ldots]$ by

$$\begin{aligned}
&[\ldots,a,x,x,y,\ldots][\ldots,y,x,x,a,\ldots][\ldots,x,x,a,y,\ldots]^{-1}[\ldots,y,a,x,x,\ldots]^{-1}\\
&[\ldots,x,x,a,y,\ldots]^{2}[\ldots,x,a,x,y,\ldots]^{-2}[\ldots,y,x,x,a,\ldots]^{-2}[\ldots,y,x,a,x,\ldots]^{2}\\
=&[\ldots,y,x,a,x,\ldots]^{3}[\ldots,x,a,x,y,\ldots]^{-3}.
\end{aligned}$$

Finally, if there is another y in a commutator then we can replace $[\ldots,[a,x,y,x],\ldots]$

by

$$
\begin{aligned}
&[\ldots,[a,x,y],x,\ldots][\ldots,x,[a,x,y],\ldots]^{-1}\\
=&[\ldots,a,x,y,x,\ldots][\ldots,x,a,y,x,\ldots]^{-1}[\ldots,y,a,x,x,\ldots]^{-1}[\ldots,y,x,a,x,\ldots]\\
&[\ldots,x,a,x,y,\ldots]^{-1}[\ldots,x,x,a,y,\ldots][\ldots,x,y,a,x,\ldots][\ldots,x,y,x,a,\ldots]^{-1}.
\end{aligned}
$$

A.4 Proofs of (i) - (v)

We now check (i) - (v). Recall that $z=[h,x,x,y,a,y,b,c]$.

Proof of (i): As commutators with a triple entry of x or y are trivial in G, the cases $t=x$, $t=[x,y]$, and $t=x[x,y]$ are trivial. Now suppose that $t=xy$. Then

$$
\begin{aligned}
[h,t,t,t]=&[h,x,x,y][h,x,y,x][h,y,x,x][h,y,y,x][h,y,x,y][h,x,y,y]\\
&[h,x,y,x,y][h,x,y,y,x][h,x,x,y,y][h,y,x,y,x][h,x,y,x,y][h,y,x,x,y]
\end{aligned}
$$

Checking (A.1) for $[h,t,t,t,a,b,c]$ we get

$$
\begin{aligned}
&[h,x,x,y,y,a,b,c]^2[h,x,y,y,x,a,b,c]^2[h,y,y,x,x,a,b,c]^2[h,y,y,x,x,a,b,c]^2\\
&[h,y,x,x,y,a,b,c]^2[h,x,x,y,y,a,b,c]^2[h,x,y,x,y,a,b,c]^4[h,x,y,y,x,a,b,c]^4\\
&[h,x,x,y,y,a,b,c]^4[h,y,x,y,x,a,b,c]^4[h,x,y,x,y,a,b,c]^4[h,y,x,x,y,a,b,c]^4\\
=&z^{-6}z^4z^{-6}z^{-6}z^4z^{-6}z^4z^8z^{-12}z^4z^4z^8=1.
\end{aligned}
$$

Commuting $[h,t,t,t]$ by y,a,b,c or a,y,b,c gives

$$
\begin{aligned}
&[h,x,x,y,y,a,b,c][h,x,y,x,y,a,b,c][h,y,x,x,y,a,b,c]=z^{-3}zz^2=1,\\
&[h,x,x,y,a,y,b,c][h,x,y,x,a,y,b,c][h,y,x,x,a,y,b,c]=zz^{-2}z=1.
\end{aligned}
$$

Commuting by a,b,y,c or a,b,c,y follows by cycling to the right four times and reflecting these two previous cases. Swapping x and y finishes the case $t=xy$. Finally consider the case $t = xy[x, y]$. Then,

$$\begin{aligned}
[h,t,t,t] =& [h,x,y,[x,y]][h,x,[x,y],y][h,y,x,[x,y]][h,y,[x,y],x] \\
& [h,[x,y],x,y][h,[x,y],y,x] \\
=& [h,x,y,x,y][h,x,y,y,x]^{-1}[h,x,x,y,y][h,x,y,x,y]^{-1}[h,y,x,x,y] \\
& [h,y,x,y,x]^{-1}[h,y,x,y,x][h,y,y,x,x]^{-1}[h,x,y,x,y][h,y,x,x,y]^{-1} \\
& [h,x,y,y,x][h,y,x,y,x]^{-1} = 1.
\end{aligned}$$

Proof of (ii): Let $P = [h,t,t,u][h,t,u,t][h,u,t,t]$. First suppose that $u = a$ and $t = x$.

$$\begin{aligned}
[h,x,x,a,y,y,b,c][h,x,a,x,y,y,b,c][h,a,x,x,y,y,b,c] =& z^2 z z^{-3} = 1, \\
[h,x,x,a,y,b,y,c][h,x,a,x,y,b,y,c][h,a,x,x,y,b,y,c] =& z^{-4} z^3 z = 1, \\
[h,x,x,a,y,b,c,y][h,x,a,x,y,b,c,y][h,a,x,x,y,b,c,y] =& z z^{-2} z = 1, \\
[h,x,x,a,b,y,y,c][h,x,a,x,b,y,y,c][h,a,x,x,b,y,y,c] =& z^2 z^{-4} z^2 = 1.
\end{aligned}$$

Cycling to the right four times and reflecting finishes this case. Now suppose that $t = [x, y]$.

$$[h,[x,y],[x,y],a,b,c][h,[x,y],a,[x,y],b,c][h,a,[x,y],[x,y],b,c] = z^{-2} z^4 z^{-2} = 1.$$

Next suppose that $u = a$ and $t = x[x, y]$. Then

$$P = [h,x,[x,y],a][h,[x,y],x,a][h,x,a,[x,y]][h,[x,y],a,x][h,a,x,[x,y]][h,a,[x,y],x].$$

Commuting by b, c and replacing $[x, y]$ using Section A.3 gives

$$\begin{aligned}
& [h,x,y,x,y,a,b,c]^{-3}[h,y,x,y,x,a,b,c]^{-3}[h,x,a,y,x,y,b,c]^{-3}[h,y,x,y,a,x,b,c]^{-3} \\
& [h,a,x,y,x,y,b,c]^{-3}[h,a,y,x,y,x,b,c]^{-3} = z^{-3} z^{-3} z^6 z^6 z^{-3} z^{-3} = 1.
\end{aligned}$$

Instead commuting on the right by y, b, c gives

$$\begin{aligned}
& [h,x,[x,y],a,y,b,c][h,[x,y],x,a,y,b,c][h,x,a,[x,y],y,b,c][h,[x,y],a,x,y,b,c] \\
& [h,a,x,[x,y],y,b,c][h,a,[x,y],x,y,b,c] = z^3 z^{-3} z^3 z^2 z^{-4} z^{-1} = 1.
\end{aligned}$$

Commuting by b, c, y follows by cycling to the right three times and reflecting and using Lemma A.1. Commuting by b, y, c gives

$$
\begin{aligned}
&[h, x, [x, y], a, b, y, c][h, [x, y], x, a, b, y, c][h, x, a, [x, y], b, y, c][h, [x, y], a, x, b, y, c] \\
&[h, a, x, [x, y], b, y, c][h, a, [x, y], x, b, y, c] = z^3 z^{-3} z^2 z^{-2} z^3 z^{-3} = 1.
\end{aligned}
$$

Next consider $u = a$ and $t = xy$. Here

$$
\begin{aligned}
P = &[h, x, y, a][h, y, x, a][h, x, a, y][h, y, a, x][h, a, x, y][h, a, y, x][h, x, y, x, a][h, x, x, y, a] \\
&[h, x, y, a, x][h, x, a, x, y][h, a, x, x, y][h, a, x, y, x][h, x, y, y, a][h, y, x, y, a][h, x, y, a, y] \\
&[h, y, a, x, y][h, a, x, y, y][h, a, y, x, y][h, x, y, x, y, a][h, x, y, a, x, y][h, a, x, y, x, y].
\end{aligned}
$$

Commuting on the right by b, c gives, in terms of (A.1),

$$
\begin{aligned}
&[h, x, x, y, y, a, b, c][h, y, y, x, x, a, b, c][h, x, x, a, y, y, b, c][h, y, y, a, x, x, b, c] \\
&[h, a, x, x, y, y, b, c][h, a, y, y, x, x, b, c][h, x, y, y, x, a, b, c]^2[h, x, x, y, y, a, b, c]^2 \\
&[h, x, y, y, a, x, b, c]^2[h, x, a, x, y, y, b, c]^2[h, a, x, x, y, y, b, c]^2[h, a, x, y, y, x, b, c]^2 \\
&[h, x, x, y, y, a, b, c]^2[h, y, x, x, y, a, b, c]^2[h, x, x, y, a, y, b, c]^2[h, y, a, x, x, y, b, c]^2 \\
&[h, a, x, x, y, y, b, c]^2[h, a, y, x, x, y, b, c]^2[h, x, y, x, y, a, b, c]^4[h, x, y, a, x, y, b, c]^4 \\
&[h, a, x, y, x, y, b, c]^4 = z^{-3} z^{-3} z^2 z^2 z^{-3} z^{-3} z^4 z^{-6} z^2 z^2 z^{-6} z^4 z^{-6} z^4 z^2 z^2 z^{-6} z^4 z^4 z^0 z^4 = 1.
\end{aligned}
$$

Instead commuting by x, b, c, by b, x, c or by b, c, x gives

$$
\begin{aligned}
&[h, x, y, y, a, x, b, c][h, y, y, x, a, x, b, c][h, x, a, y, y, x, b, c][h, y, y, a, x, x, b, c] \\
&[h, a, x, y, y, x, b, c][h, a, y, y, x, x, b, c][h, x, y, y, a, x, b, c]^2[h, y, x, y, a, x, b, c]^2 \\
&[h, x, y, a, y, x, b, c]^2[h, y, a, x, y, x, b, c]^2[h, a, x, y, y, x, b, c]^2[h, a, y, x, y, x, b, c]^2 \\
=&zzzz^2 z^2 z^{-3} z^2 z^{-4} z^{-4} z^{-4} z^4 z^2 = 1,
\end{aligned}
$$

$$
\begin{aligned}
&[h, x, y, y, a, b, x, c][h, y, y, x, a, b, x, c][h, x, a, y, y, b, x, c][h, y, y, a, x, b, x, c] \\
&[h, a, x, y, y, b, x, c][h, a, y, y, x, b, x, c][h, x, y, y, a, b, x, c]^2[h, y, x, y, a, b, x, c]^2 \\
&[h, x, y, a, y, b, x, c]^2[h, y, a, x, y, b, x, c]^2[h, a, x, y, y, b, x, c]^2[h, a, y, x, y, b, x, c]^2 \\
=&zzz^{-4} z^{-4} zzz^2 z^{-4} z^6 z^2 z^2 z^{-4} = 1,
\end{aligned}
$$

$$
\begin{aligned}
&[h,x,y,y,a,b,c,x][h,y,y,x,a,b,c,x][h,x,a,y,y,b,c,x][h,y,y,a,x,b,c,x]\\
&[h,a,x,y,y,b,c,x][h,a,y,y,x,b,c,x][h,x,y,y,a,b,c,x]^2[h,y,x,y,a,b,c,x]^2\\
&[h,x,y,a,y,b,c,x]^2[h,y,a,x,y,b,c,x]^2[h,a,x,y,y,b,c,x]^2[h,a,y,x,y,b,c,x]^2\\
=&z^{-3}z^2z^2zzzz^{-6}z^2z^2z^0z^2z^{-4}=1.
\end{aligned}
$$

Note that commuting by y,b,c, by b,y,c or by b,c,y follows by cycling to the right three times, reflecting and swapping x and y in the previous three cases above. To finish the case $u=a$ and $t=xy$, by symmetry of x and y in the weight 4 terms from P, the reflecting property and cycling we just check

$$
\begin{aligned}
&[h,x,y,a,x,y,b,c][h,y,x,a,x,y,b,c][h,x,a,y,x,y,b,c][h,y,a,x,x,y,b,c]\\
&[h,a,x,y,x,y,b,c][h,a,y,x,x,y,b,c]=z^0z^{-2}z^{-2}zzz^2=1,\\
&[h,x,y,a,x,b,y,c][h,y,x,a,x,b,y,c][h,x,a,y,x,b,y,c][h,y,a,x,x,b,y,c]\\
&[h,a,x,y,x,b,y,c][h,a,y,x,x,b,y,c]=zz^3zz^{-4}z^{-2}z=1,\\
&[h,x,y,a,x,b,c,y][h,y,x,a,x,b,c,y][h,x,a,y,x,b,c,y][h,y,a,x,x,b,c,y]\\
&[h,a,x,y,x,b,c,y][h,a,y,x,x,b,c,y]=z^{-2}zz^0z^2z^{-2}z=1,\\
&[h,x,y,a,b,x,y,c][h,y,x,a,b,x,y,c][h,x,a,y,b,x,y,c][h,y,a,x,b,x,y,c]\\
&[h,a,x,y,b,x,y,c][h,a,y,x,b,x,y,c]=z^0z^{-2}zz^3z^0z^{-2}=1.
\end{aligned}
$$

Lastly for $u=a$ we check the case $t=xy[x,y]$. Note that replacing one t by x and one by $y,[x,y]$ cancels with replacing one by y and the other by $x,[x,y]$. Hence commuting P by b,c gives

$$
\begin{aligned}
&[h,x,y,[x,y],a,b,c][h,[x,y],x,y,a,b,c][h,x,y,a,[x,y],b,c][h,[x,y],a,x,y,b,c]\\
&[h,a,x,y,[x,y],b,c][h,a,[x,y],x,y,b,c]=z^{-1}z^{-1}z^2z^2z^{-1}z^{-1}=1.
\end{aligned}
$$

Next suppose that $u=[a,x]$. The only non-trivial case to check is $t=xy$. Here

$$
\begin{aligned}
P=&[h,x,y,[a,x]][h,y,x,[a,x]][h,x,[a,x],y][h,y,[a,x],x][h,[a,x],x,y]\\
&[h,[a,x],y,x][h,x,y,y,[a,x]][h,y,x,y,[a,x]][h,x,y,[a,x],y][h,y,[a,x],x,y]\\
&[h,[a,x],x,y,y][h,[a,x],y,x,y].
\end{aligned}
$$

Commuting on the right by b, c gives, in terms of (A.1),

$$\begin{aligned}&[h,x,y,y,[a,x],b,c][h,y,y,x,[a,x],b,c][h,x,[a,x],y,y,b,c][h,y,y,[a,x],x,b,c]\\&[h,[a,x],x,y,y,b,c][h,[a,x],y,y,x,b,c][h,x,y,y,[a,x],b,c]^2[h,y,x,y,[a,x],b,c]^2\\&[h,x,y,[a,x],y,b,c]^2[h,y,[a,x],x,y,b,c]^2[h,[a,x],x,y,y,b,c]^2[h,[a,x],y,x,y,b,c]^2\\&=z^{-1}z^4z^{-1}zz^{-4}zz^{-2}z^{-6}z^4z^6z^{-8}z^6=1.\end{aligned}$$

Instead commuting by y, b, c gives

$$\begin{aligned}&[h,x,y,[a,x],y,b,c][h,y,x,[a,x],y,b,c][h,x,[a,x],y,y,b,c][h,y,[a,x],x,y,b,c]\\&[h,[a,x],x,y,y,b,c][h,[a,x],y,x,y,b,c]=z^2z^{-3}z^{-1}z^3z^{-4}z^3=1.\end{aligned}$$

Commuting by b, c, y follows by cycling to the right three times and reflecting and using Lemma A.1. Commuting by b, y, c gives

$$\begin{aligned}&[h,x,y,[a,x],b,y,c][h,y,x,[a,x],b,y,c][h,x,[a,x],y,b,y,c][h,y,[a,x],x,b,y,c]\\&[h,[a,x],x,y,b,y,c][h,[a,x],y,x,b,y,c]=z^3z^2z^7z^{-7}z^{-2}z^{-3}=1.\end{aligned}$$

Next suppose that $u = [a, y]$ and $t = x$. Then

$$P=[h,x,x,[a,y]][h,x,[a,y],x][h,[a,y],x,x].$$

Commuting by b, c and using Section A.3 gives

$$[h,x,x,y,a,y,b,c]^{-3}[h,x,y,a,y,x,b,c]^{-3}[h,y,a,y,x,x,b,c]^{-3}=z^{-3}z^6z^{-3}=1.$$

Instead commuting by y, b, c gives

$$[h,x,x,[a,y],y,b,c][h,x,[a,y],x,y,b,c][h,[a,y],x,x,y,b,c]=zz^{-2}z=1.$$

Commuting by b, c, y follows by cycling to the right three times and reflecting and using Lemma A.1. Commuting by b, y, c gives

$$[h,x,x,[a,y],b,y,c][h,x,[a,y],x,b,y,c][h,[a,y],x,x,b,y,c]=z^{-5}z^0z^5=1.$$

Next suppose that $u = [a, y]$ and $t = xy$. Then

$$\begin{aligned}P =&[h, x, y, [a, y]][h, y, x, [a, y]][h, x, [a, y], y][h, y, [a, y], x][h, [a, y], x, y]\\&[h, [a, y], y, x][h, x, y, x, [a, y]][h, x, x, y, [a, y]][h, x, y, [a, y], x]\\&[h, x, [a, y], x, y][h, [a, y], x, y, x]h, [a, y], x, x, y].\end{aligned}$$

Commuting on the right by b, c gives, in terms of (A.1),

$$\begin{aligned}&[h, x, x, y, [a, y], b, c][h, y, x, x, [a, y], b, c][h, x, x, [a, y], y, b, c][h, y, [a, y], x, x, b, c]\\&[h, [a, y], x, x, y, b, c][h, [a, y], y, x, x, b, c][h, x, y, x, [a, y], b, c]^2[h, x, x, y, [a, y], b, c]^2\\&[h, x, y, [a, y], x, b, c]^2[h, x, [a, y], x, y, b, c]^2[h, [a, y], x, y, x, b, c]^2[h, [a, y], x, x, y, b, c]^2\\&= z^4z^{-1}zz^{-1}zz^{-4}z^{-6}z^8z^{-6}z^{-4}z^6z^2 = 1.\end{aligned}$$

Commuting by y, b, c, b, y, c or b, c, y satisfies (A.1) by swapping x and y in the case $u = [a, x]$ and $t = xy$. Finally for $u = [a, y]$ suppose that $t = x[x, y]$. Then

$$\begin{aligned}P =&[h, x, [x, y], [a, y]][h, [x, y], x, [a, y]][h, x, [a, y], [x, y]]\\&[h, [x, y], [a, y], x][h, [a, y], x, [x, y]][h, [a, y], [x, y], x].\end{aligned}$$

Commuting on the right by b, c gives

$$\begin{aligned}&[h, x, [x, y], [a, y], b, c][h, [x, y], x, [a, y], b, c][h, x, [a, y], [x, y], b, c][h, [x, y], [a, y], x, b, c]\\&[h, [a, y], x, [x, y], b, c][h, [a, y], [x, y], x, b, c] = z^7z^{-2}z^{-5}z^{-5}z^{-2}z^7 = 1.\end{aligned}$$

Next suppose that $u = [a, x, y]$. The only case to consider here is $t = xy$. In this case commuting P on the right by b, c gives

$$\begin{aligned}&[h, x, y, [a, x, y], b, c][h, y, x, [a, x, y], b, c][h, x, [a, x, y], y, b, c][h, y, [a, x, y], x, b, c]\\&[h, [a, x, y], x, y, b, c][h, [a, x, y], y, x, b, c] = z^3z^0z^{-3}z^{-3}z^0z^3 = 1.\end{aligned}$$

By the symmetry in P swapping x and y gives the case $u = [a, y, x]$, which completes the proof of (ii).

Proof of (iii): Let

$$\begin{aligned}Q =&[h, t, u, v][h, u, t, v][h, t, v, u][h, u, v, t][h, v, t, u][h, v, u, t]\\&[h, u, t, t, v]^{-1}[h, t, u, v, t][h, t, v, t, u][h, v, u, t, t]^{-1}.\end{aligned}$$

First suppose that u is the same as v, but with b replaced by a. Then we have that commutators are alternating in u and v. Hence Q becomes

$$[h,u,t,t,v]^{-1}[h,t,u,v,t][h,t,u,t,v]^{-1}[h,u,v,t,t].$$

Suppose that $u=a$ and $v=b$. For $t=x$ we get

$$Q=[h,a,x,x,b]^{-1}[h,x,a,b,x][h,x,a,x,b]^{-1}[h,a,b,x,x].$$

We have that

$$\begin{aligned}
&[h,a,x,x,b,y,y,c]^{-1}[h,x,a,b,x,y,y,c][h,x,a,x,b,y,y,c]^{-1}\\
&[h,a,b,x,x,y,y,c]=z^{-2}zz^{4}z^{-3}=1,\\
&[h,a,x,x,b,y,c,y]^{-1}[h,x,a,b,x,y,c,y][h,x,a,x,b,y,c,y]^{-1}\\
&[h,a,b,x,x,y,c,y]=z^{4}z^{-2}z^{-3}z=1,\\
&[h,a,x,x,b,c,y,y]^{-1}[h,x,a,b,x,c,y,y][h,x,a,x,b,c,y,y]^{-1}\\
&[h,a,b,x,x,c,y,y]=z^{-2}zz^{-1}z^{2}=1.
\end{aligned}$$

Next suppose that $t=[x,y]$. Commuting Q by c gives

$$\begin{aligned}
&[h,a,[x,y],[x,y],b,c]^{-1}[h,[x,y],a,b,[x,y],c][h,[x,y],a,[x,y],b,c]^{-1}\\
&[h,a,b,[x,y],[x,y],c]=z^{2}z^{4}z^{-4}z^{-2}=1.
\end{aligned}$$

When $t=xy$ we get

$$\begin{aligned}
Q=&[h,a,x,y,b]^{-1}[h,a,y,x,b]^{-1}[h,x,a,b,y][h,y,a,b,x][h,x,a,y,b]^{-1}[h,y,a,x,b]^{-1}\\
&[h,a,b,x,y][h,a,b,y,x][h,a,x,x,y,b]^{-1}[h,a,x,y,x,b]^{-1}[h,x,a,b,x,y]\\
&[h,x,y,a,b,x][h,x,a,x,y,b]^{-1}[h,x,y,a,x,b]^{-1}[h,a,b,x,x,y][h,a,b,x,y,x]\\
&[h,a,x,y,y,b]^{-1}[h,a,y,x,y,b]^{-1}[h,x,y,a,b,y][h,y,a,b,x,y][h,x,y,a,y,b]^{-1}\\
&[h,y,a,x,y,b]^{-1}[h,a,b,x,y,y][h,a,b,y,x,y][h,a,x,y,x,y,b]^{-1}[h,x,y,a,b,x,y]\\
&[h,x,y,a,x,y,b]^{-1}[h,a,b,x,y,x,y].
\end{aligned}$$

Commuting on the right by c gives, in terms of (A.1),

$$\begin{aligned}
&[h,a,x,x,y,y,b,c]^{-1}[h,a,y,y,x,x,b,c]^{-1}[h,x,x,a,b,y,y,c][h,y,y,a,b,x,x,c]\\
&[h,x,x,a,y,y,b,c]^{-1}[h,y,y,a,x,x,b,c]^{-1}[h,a,b,x,x,y,y,c][h,a,b,y,y,x,x,c]\\
&([h,a,x,x,y,y,b,c]^{-1}[h,a,x,y,y,x,b,c]^{-1}[h,x,a,b,x,y,y,c][h,x,y,y,a,b,x,c]\\
&[h,x,a,x,y,y,b,c]^{-1}[h,x,y,y,a,x,b,c]^{-1}[h,a,b,x,x,y,y,c][h,a,b,x,y,y,x,c]\\
&[h,a,x,x,y,y,b,c]^{-1}[h,a,y,x,x,y,b,c]^{-1}[h,x,x,y,a,b,y,c][h,y,a,b,x,x,y,c]\\
&[h,x,x,y,a,y,b,c]^{-1}[h,y,a,x,x,y,b,c]^{-1}[h,a,b,x,x,y,y,c][h,a,b,y,x,x,y,c])^2\\
&([h,a,x,y,x,y,b,c]^{-1}[h,x,y,a,b,x,y,c][h,x,y,a,x,y,b,c]^{-1}[h,a,b,x,y,x,y,c])^4\\
=&z^3z^3z^2z^2z^{-2}z^{-2}z^{-3}z^{-3}(z^3z^{-2}zzz^{-1}z^{-1}z^{-3}z^2z^3z^{-2}zzz^{-1}z^{-1}z^{-3}z^2)^2(z^{-1}z^0z^0z)^4=1.
\end{aligned}$$

Commuting instead by x,c or by c,x gives

$$\begin{aligned}
&[h,a,x,y,y,b,x,c]^{-1}[h,a,y,y,x,b,x,c]^{-1}[h,x,a,b,y,y,x,c][h,y,y,a,b,x,x,c]\\
&[h,x,a,y,y,b,x,c]^{-1}[h,y,y,a,x,b,x,c]^{-1}[h,a,b,x,y,y,x,c][h,a,b,y,y,x,x,c]\\
&[h,a,x,y,y,b,x,c]^{-2}[h,a,y,x,y,b,x,c]^{-2}[h,x,y,a,b,y,x,c]^2[h,y,a,b,x,y,x,c]^2\\
&[h,x,y,a,y,b,x,c]^{-2}[h,y,a,x,y,b,x,c]^{-2}[h,a,b,x,y,y,x,c]^2[h,a,b,y,x,y,x,c]^2\\
=&z^{-1}z^{-1}zz^2z^4z^4z^2z^{-3}z^{-2}z^4z^{-4}z^{-4}z^{-6}z^{-2}z^4z^2=1,
\end{aligned}$$

$$\begin{aligned}
&[h,a,x,y,y,b,c,x]^{-1}[h,a,y,y,x,b,c,x]^{-1}[h,x,a,b,y,y,c,x][h,y,y,a,b,x,c,x]\\
&[h,x,a,y,y,b,c,x]^{-1}[h,y,y,a,x,b,c,x]^{-1}[h,a,b,x,y,y,c,x][h,a,b,y,y,x,c,x]\\
&[h,a,x,y,y,b,c,x]^{-2}[h,a,y,x,y,b,c,x]^{-2}[h,x,y,a,b,y,c,x]^2[h,y,a,b,x,y,c,x]^2\\
&[h,x,y,a,y,b,c,x]^{-2}[h,y,a,x,y,b,c,x]^{-2}[h,a,b,x,y,y,c,x]^2[h,a,b,y,x,y,c,x]^2\\
=&z^{-1}z^{-1}z^2zz^{-2}z^{-1}zzz^{-2}z^4z^2z^0z^{-2}z^0z^2z^{-4}=1.
\end{aligned}$$

Commuting by y,c or by c,y gives

$$\begin{aligned}
&[h,a,x,x,y,b,y,c]^{-1}[h,a,y,x,x,b,y,c]^{-1}[h,x,x,a,b,y,y,c][h,y,a,b,x,x,y,c]\\
&[h,x,x,a,y,b,y,c]^{-1}[h,y,a,x,x,b,y,c]^{-1}[h,a,b,x,x,y,y,c][h,a,b,y,x,x,y,c]\\
&[h,a,x,x,y,b,y,c]^{-2}[h,a,x,y,x,b,y,c]^{-2}[h,x,a,b,x,y,y,c]^2[h,x,y,a,b,x,y,c]^2\\
&[h,x,a,x,y,b,y,c]^{-2}[h,x,y,a,x,b,y,c]^{-2}[h,a,b,x,x,y,y,c]^2[h,a,b,x,y,x,y,c]^2\\
=&z^{-1}z^{-1}z^2zz^4z^4z^{-3}z^2z^{-2}z^4z^2z^0z^{-6}z^{-2}z^{-6}z^2=1,
\end{aligned}$$

$$
\begin{aligned}
&[h,a,x,x,y,b,c,y]^{-1}[h,a,y,x,x,b,c,y]^{-1}[h,x,x,a,b,y,c,y][h,y,a,b,x,x,c,y]\\
&[h,x,x,a,y,b,c,y]^{-1}[h,y,a,x,x,b,c,y]^{-1}[h,a,b,x,x,y,c,y][h,a,b,y,x,x,c,y]\\
&[h,a,x,x,y,b,c,y]^{-2}[h,a,x,y,x,b,c,y]^{-2}[h,x,a,b,x,y,c,y]^{2}[h,x,y,a,b,x,c,y]^{2}\\
&[h,x,a,x,y,b,c,y]^{-2}[h,x,y,a,x,b,c,y]^{-2}[h,a,b,x,x,y,c,y]^{2}[h,a,b,x,y,x,c,y]^{2}\\
=&z^{-1}z^{-1}zz^{2}z^{-1}z^{-2}zzz^{-2}z^{4}z^{-4}z^{-4}z^{4}z^{4}z^{2}z^{-4}=1.
\end{aligned}
$$

Commuting instead by x, y, c, by x, c, y or by c, x, y gives

$$
\begin{aligned}
&[h,a,x,y,b,x,y,c]^{-1}[h,a,y,x,b,x,y,c]^{-1}[h,x,a,b,y,x,y,c][h,y,a,b,x,x,y,c]\\
&[h,x,a,y,b,x,y,c]^{-1}[h,y,a,x,b,x,y,c]^{-1}[h,a,b,x,y,x,y,c][h,a,b,y,x,x,y,c]\\
=&z^{0}z^{2}z^{-2}zz^{-1}z^{-3}zz^{2}=1,
\end{aligned}
$$

$$
\begin{aligned}
&[h,a,x,y,b,x,c,y]^{-1}[h,a,y,x,b,x,c,y]^{-1}[h,x,a,b,y,x,c,y][h,y,a,b,x,x,c,y]\\
&[h,x,a,y,b,x,c,y]^{-1}[h,y,a,x,b,x,c,y]^{-1}[h,a,b,x,y,x,c,y][h,a,b,y,x,x,c,y]\\
=&z^{-1}z^{-3}z^{0}z^{2}z^{-1}z^{4}z^{-2}z=1,
\end{aligned}
$$

$$
\begin{aligned}
&[h,a,x,y,b,c,x,y]^{-1}[h,a,y,x,b,c,x,y]^{-1}[h,x,a,b,y,c,x,y][h,y,a,b,x,c,x,y]\\
&[h,x,a,y,b,c,x,y]^{-1}[h,y,a,x,b,c,x,y]^{-1}[h,a,b,x,y,c,x,y][h,a,b,y,x,c,x,y]\\
=&z^{0}z^{2}z^{-2}zz^{2}z^{-1}z^{0}z^{-2}=1.
\end{aligned}
$$

Commuting by y, x, c, by y, c, x or by c, y, x follows by swapping x and y. Next consider $t = x[x, y]$. Here

$$
\begin{aligned}
Q=&[h,a,x,[x,y],b]^{-1}[h,a,[x,y],x,b]^{-1}[h,x,a,b,[x,y]][h,[x,y],a,b,x]\\
&[h,x,a,[x,y],b]^{-1}[h,[x,y],a,x,b]^{-1}[h,a,b,x,[x,y]][h,a,b,[x,y],x].
\end{aligned}
$$

Commuting by y, c or by c, y gives

$$
\begin{aligned}
&[h,a,x,[x,y],b,y,c]^{-1}[h,a,[x,y],x,b,y,c]^{-1}[h,x,a,b,[x,y],y,c][h,[x,y],a,b,x,y,c]\\
&[h,x,a,[x,y],b,y,c]^{-1}[h,[x,y],a,x,b,y,c]^{-1}[h,a,b,x,[x,y],y,c][h,a,b,[x,y],x,y,c]\\
=&z^{-3}z^{3}z^{3}z^{2}z^{-2}z^{2}z^{-4}z^{-1}=1,
\end{aligned}
$$

$$
\begin{aligned}
&[h,a,x,[x,y],b,c,y]^{-1}[h,a,[x,y],x,b,c,y]^{-1}[h,x,a,b,[x,y],c,y][h,[x,y],a,b,x,c,y]\\
&[h,x,a,[x,y],b,c,y]^{-1}[h,[x,y],a,x,b,c,y]^{-1}[h,a,b,x,[x,y],c,y][h,a,b,[x,y],x,c,y]\\
=&z^{-3}z^{3}z^{-2}z^{-3}z^{2}z^{3}z^{3}z^{-3}=1.
\end{aligned}
$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$\begin{aligned}&[h,a,x,y,x,y,b,c]^3[h,a,y,x,y,x,b,c]^3[h,x,a,b,y,x,y,c]^{-3}[h,y,x,y,a,b,x,c]^{-3}\\&[h,x,a,y,x,y,b,c]^3[h,y,x,y,a,x,b,c]^3[h,a,b,x,y,x,y,c]^{-3}[h,a,b,y,x,y,x,c]^{-3}\\=&z^3z^3z^6z^6z^{-6}z^{-6}z^{-3}z^{-3}=1.\end{aligned}$$

Finally for $u=a$ and $v=b$ consider $t=xy[x,y]$. Replacing one t by x and one by y, $[x,y]$ cancels with replacing one by y and the other by x, $[x,y]$. Hence, commuting Q by c we get

$$\begin{aligned}&[h,a,x,y,[x,y],b,c]^{-1}[h,a,[x,y],x,y,b,c]^{-1}[h,x,y,a,b,[x,y],c][h,[x,y],a,b,x,y,c]\\&[h,x,y,a,[x,y],b,c]^{-1}[h,[x,y],a,x,y,b,c]^{-1}[h,a,b,x,y,[x,y],c][h,a,b,[x,y],x,y,c]\\=&zzz^{-2}z^{-2}z^2z^2z^{-1}z^{-1}=1.\end{aligned}$$

Next suppose that $u=[a,y]$ and $v=[b,y]$. The only case to check here is $t=x$. Then, commuting Q by c we get

$$\begin{aligned}&[h,[a,y],x,x,[b,y],c]^{-1}[h,x,[a,y],[b,y],x,c][h,x,[a,y],x,[b,y],c]^{-1}\\&[h,[a,y],[b,y],x,x,c]=z^{-4}z^{12}z^{-2}z^{-6}=1.\end{aligned}$$

Now suppose that u is not v with b replaced by a. First suppose that $u=a$ and $v=[a,x]$. Suppose that $t=x$. Commuting Q by y,y,c gives

$$\begin{aligned}&[h,x,a,[b,x],y,y,c][h,a,x,[b,x],y,y,c][h,x,[b,x],a,y,y,c][h,a,[b,x],x,y,y,c]\\&[h,[b,x],x,a,y,y,c][h,[b,x],a,x,y,y,c]=z^5z^{-1}z^6z^{-4}z^{-6}z^0=1.\end{aligned}$$

Commuting by c,y,y follows by cycling to the right three times, reflecting and using Lemma A.1. Finally, commuting by y,c,y gives

$$\begin{aligned}&[h,x,a,[b,x],y,c,y][h,a,x,[b,x],y,c,y][h,x,[b,x],a,y,c,y][h,a,[b,x],x,y,c,y]\\&[h,[b,x],x,a,y,c,y][h,[b,x],a,x,y,c,y]=z^{-5}z^7z^{-2}z^{-2}z^7z^{-5}=1.\end{aligned}$$

Next suppose instead that $t=[x,y]$. Then

$$\begin{aligned}Q=&[h,[x,y],a,[b,x]][h,a,[x,y],[b,x]][h,[x,y],[b,x],a][h,a,[b,x],[x,y]]\\&[h,[b,x],[x,y],a][h,[b,x],a,[x,y]].\end{aligned}$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$([h,y,x,y,a,[b,x],c][h,a,y,x,y,[b,x],c][h,y,x,y,[b,x],a,c][h,a,[b,x],y,x,y,c]$$
$$[h,[b,x],y,x,y,a,c][h,[b,x],a,y,x,y,c])^{-3}=(z^0z^{-3}z^3z^3z^{-3}z^0)^{-3}=1.$$

Commuting instead by y,c gives

$$[h,[x,y],a,[b,x],y,c][h,a,[x,y],[b,x],y,c][h,[x,y],[b,x],a,y,c][h,a,[b,x],[x,y],y,c]$$
$$[h,[b,x],[x,y],a,y,c][h,[b,x],a,[x,y],y,c]=z^4z^5z^{-1}z^{-7}z^{-1}z^0=1.$$

Commuting by c,y follows by cycling twice to the right, reflecting and using Lemma A.1. Now suppose that $t=xy$. Then

$$\begin{aligned}Q=&[h,x,y,a,[b,x]][h,a,x,y,[b,x]][h,x,y,[b,x],a][h,a,[b,x],x,y][h,[b,x],x,y,a]\\&[h,[b,x],a,x,y][h,a,x,y,[b,x]]^{-1}[h,a,y,x,[b,x]]^{-1}[h,x,a,[b,x],y][h,y,a,[b,x],x]\\&[h,x,[b,x],y,a][h,y,[b,x],x,a][h,[b,x],a,x,y]^{-1}[h,[b,x],a,y,x]^{-1}\\&[h,a,x,y,y,[b,x]]^{-1}[h,a,y,x,y,[b,x]]^{-1}[h,x,y,a,[b,x],y][h,y,a,[b,x],x,y]\\&[h,x,y,[b,x],y,a][h,y,[b,x],x,y,a][h,[b,x],a,x,y,y]^{-1}[h,[b,x],a,y,x,y]^{-1}.\end{aligned}$$

Commuting by c gives, in terms of (A.1),

$$\begin{aligned}&[h,x,y,y,a,[b,x],c][h,a,x,y,y,[b,x],c][h,x,y,y,[b,x],a,c][h,a,[b,x],x,y,y,c]\\&[h,[b,x],x,y,y,a,c][h,[b,x],a,x,y,y,c][h,a,x,y,y,[b,x],c]^{-1}[h,a,y,y,x,[b,x],c]^{-1}\\&[h,x,a,[b,x],y,y,c][h,y,y,a,[b,x],x,c][h,x,[b,x],y,y,a,c][h,y,y,[b,x],x,a,c]\\&[h,[b,x],a,x,y,y,c]^{-1}[h,[b,x],a,y,y,x,c]^{-1}[h,a,x,y,y,[b,x],c]^{-2}\\&[h,a,y,x,y,[b,x],c]^{-2}[h,x,y,a,[b,x],y,c]^2[h,y,a,[b,x],x,y,c]^2[h,x,y,[b,x],y,a,c]^2\\&[h,y,[b,x],x,y,a,c]^2[h,[b,x],a,x,y,y,c]^{-2}[h,[b,x],a,y,x,y,c]^{-2}\\&=z^0z^{-1}zz^{-4}z^4z^0zz^{-4}z^5z^6zz^{-1}z^0z^0z^2z^6z^{-2}z^{-4}z^{-4}z^{-6}z^0z^0=1.\end{aligned}$$

Commuting instead by y,c or by c,y gives

$$\begin{aligned}&[h,x,y,a,[b,x],y,c][h,a,x,y,[b,x],y,c][h,x,y,[b,x],a,y,c][h,a,[b,x],x,y,y,c]\\&[h,[b,x],x,y,a,y,c][h,[b,x],a,x,y,y,c][h,a,x,y,[b,x],y,c]^{-1}[h,a,y,x,[b,x],y,c]^{-1}\\&[h,x,a,[b,x],y,y,c][h,y,a,[b,x],x,y,c][h,x,[b,x],y,a,y,c][h,y,[b,x],x,a,y,c]\\&[h,[b,x],a,x,y,y,c]^{-1}[h,[b,x],a,y,x,y,c]^{-1}\\&=z^{-1}z^2z^{-3}z^{-4}z^2z^0z^{-2}z^3z^5z^{-2}z^{-7}z^7z^0z^0=1,\end{aligned}$$

$$[h,x,y,a,[b,x],c,y][h,a,x,y,[b,x],c,y][h,x,y,[b,x],a,c,y][h,a,[b,x],x,y,c,y]$$
$$[h,[b,x],x,y,a,c,y][h,[b,x],a,x,y,c,y][h,a,x,y,[b,x],c,y]^{-1}[h,a,y,x,[b,x],c,y]^{-1}$$
$$[h,x,a,[b,x],y,c,y][h,y,a,[b,x],x,c,y][h,x,[b,x],y,a,c,y][h,y,[b,x],x,a,c,y]$$
$$[h,[b,x],a,x,y,c,y]^{-1}[h,[b,x],a,y,x,c,y]^{-1}$$
$$= z^0z^3z^3z^{-2}z^{-3}z^{-5}z^{-3}z^{-2}z^{-5}z^6z^3z^{-1}z^5z = 1.$$

Next suppose that $u = a$ and $v = [b,y]$. The case $t = [x,y]$ follows from the above by swapping x and y. Consider $t = x$. Then

$$Q = [h,x,a,[b,y]][h,a,x,[b,y]][h,x,[b,y],a][h,a,[b,y],x][h,[b,y],x,a][h,[b,y],a,x]$$
$$[h,a,x,x,[b,y]]^{-1}[h,x,a,[b,y],x][h,x,[b,y],x,a][h,[b,y],a,x,x]^{-1}.$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$([h,x,x,a,y,b,y,c][h,a,x,x,y,b,y,c][h,x,x,y,b,y,a,c][h,a,y,b,y,x,x,c]$$
$$[h,y,b,y,x,x,a,c][h,y,b,y,a,x,x,c][h,a,x,x,y,b,y,c]^{-2}[h,x,a,y,b,y,x,c]^2$$
$$[h,x,y,b,y,x,a,c]^2[h,y,b,y,a,x,x,c]^{-2})^{-3} = (z^{-4}zz^{-1}zz^{-1}z^4z^{-2}z^6z^4z^{-8})^{-3} = 1.$$

Commuting instead by y,c or by c,y gives

$$[h,x,x,a,[b,y],y,c][h,a,x,x,[b,y],y,c][h,x,x,[b,y],a,y,c][h,a,[b,y],x,x,y,c]$$
$$[h,[b,y],x,x,a,y,c][h,[b,y],a,x,x,y,c][h,a,x,x,[b,y],y,c]^{-2}[h,x,a,[b,y],x,y,c]^2$$
$$[h,x,[b,y],x,a,y,c]^2[h,[b,y],a,x,x,y,c]^{-2} = z^6zz^5zz^{-5}z^0z^{-2}z^{-6}z^0z^0 = 1,$$

$$[h,x,x,a,[b,y],c,y][h,a,x,x,[b,y],c,y][h,x,x,[b,y],a,c,y][h,a,[b,y],x,x,c,y]$$
$$[h,[b,y],x,x,a,c,y][h,[b,y],a,x,x,c,y][h,a,x,x,[b,y],c,y]^{-2}[h,x,a,[b,y],x,c,y]^2$$
$$[h,x,[b,y],x,a,c,y]^2[h,[b,y],a,x,x,c,y]^{-2} = z^0z^{-5}zz^5zz^6z^{10}z^{-2}z^{-4}z^{-12} = 1.$$

Commuting by x,c and using Section A.3 gives

$$([h,x,a,y,b,y,x,c][h,a,x,y,b,y,x,c][h,x,y,b,y,a,x,c][h,a,y,b,y,x,x,c]$$
$$[h,y,b,y,x,a,x,c][h,y,b,y,a,x,x,c])^{-3} = (z^3z^{-2}z^{-3}zz^{-3}z^4)^{-3} = 1.$$

Commuting by c,x follows by cycling twice to the right and reflecting, which gives the previous case. Commuting by x, y, c, by x, c, y or by c, x, y gives

$$[h, x, a, [b, y], x, y, c][h, a, x, [b, y], x, y, c][h, x, [b, y], a, x, y, c][h, a, [b, y], x, x, y, c]$$
$$[h, [b, y], x, a, x, y, c][h, [b, y], a, x, x, y, c] = z^{-3}z^{-2}z^{-1}zz^5z^0 = 1,$$

$$[h, x, a, [b, y], x, c, y][h, a, x, [b, y], x, c, y][h, x, [b, y], a, x, c, y][h, a, [b, y], x, x, c, y]$$
$$[h, [b, y], x, a, x, c, y][h, [b, y], a, x, x, c, y] = z^{-1}z^0z^{-3}z^5z^{-7}z^6 = 1,$$

$$[h, x, a, [b, y], c, x, y][h, a, x, [b, y], c, x, y][h, x, [b, y], a, c, x, y][h, a, [b, y], x, c, x, y]$$
$$[h, [b, y], x, a, c, x, y][h, [b, y], a, x, c, x, y] = z^0zz^3z^{-5}z^3z^{-2} = 1.$$

Commuting by y, x, c, by y, c, x or by c, y, x follows by cycling to the right three times and then reflecting, which gives the previous three cases. Now suppose that $t = xy$. Then

$$\begin{aligned}Q =&[h, x, y, a, [b, y]][h, a, x, y, [b, y]][h, x, y, [b, y], a][h, a, [b, y], x, y][h, [b, y], x, y, a]\\
&[h, [b, y], a, x, y][h, a, x, y, [b, y]]^{-1}[h, a, y, x, [b, y]]^{-1}[h, x, a, [b, y], y][h, y, a, [b, y], x]\\
&[h, x, [b, y], y, a][h, y, [b, y], x, a][h, [b, y], a, x, y]^{-1}[h, [b, y], a, y, x]^{-1}\\
&[h, a, x, x, y, [b, y]]^{-1}[h, a, x, y, x, [b, y]]^{-1}[h, x, a, [b, y], x, y][h, x, y, a, [b, y], x]\\
&[h, x, [b, y], x, y, a][h, x, y, [b, y], x, a][h, [b, y], a, x, x, y]^{-1}[h, [b, y], a, x, y, x]^{-1}.\end{aligned}$$

Commuting on the right by c gives, in terms of (A.1),

$$\begin{aligned}&[h, x, x, y, a, [b, y], c][h, a, x, x, y, [b, y], c][h, x, x, y, [b, y], a, c][h, a, [b, y], x, x, y, c]\\
&[h, [b, y], x, x, y, a, c][h, [b, y], a, x, x, y, c][h, a, x, x, y, [b, y], c]^{-1}[h, a, y, x, x, [b, y], c]^{-1}\\
&[h, x, x, a, [b, y], y, c][h, y, a, [b, y], x, x, c][h, x, x, [b, y], y, a, c][h, y, [b, y], x, x, a, c]\\
&[h, [b, y], a, x, x, y, c]^{-1}[h, [b, y], a, y, x, x, c]^{-1}[h, a, x, x, y, [b, y], c]^{-2}\\
&[h, a, x, y, x, [b, y], c]^{-2}[h, x, a, [b, y], x, y, c]^2[h, x, y, a, [b, y], x, c]^2[h, x, [b, y], x, y, a, c]^2\\
&[h, x, y, [b, y], x, a, c]^2[h, [b, y], a, x, x, y, c]^{-2}[h, [b, y], a, x, y, x, c]^{-2}\\
=&z^0z^4z^{-4}zz^{-1}z^0z^{-4}zz^6z^5z^{-1}zz^0z^0z^{-8}z^6z^{-6}z^{-10}z^4z^6z^0z^0 = 1.\end{aligned}$$

Commuting instead by x, c or by c, x gives

$$\begin{aligned}
&[h,x,y,a,[b,y],x,c][h,a,x,y,[b,y],x,c][h,x,y,[b,y],a,x,c][h,a,[b,y],x,y,x,c]\\
&[h,[b,y],x,y,a,x,c][h,[b,y],a,x,y,x,c][h,a,x,y,[b,y],x,c]^{-1}[h,a,y,x,[b,y],x,c]^{-1}\\
&[h,x,a,[b,y],y,x,c][h,y,a,[b,y],x,x,c][h,x,[b,y],y,a,x,c][h,y,[b,y],x,a,x,c]\\
&[h,[b,y],a,x,y,x,c]^{-1}[h,[b,y],a,y,x,x,c]^{-1}\\
=&z^{-5}z^{-3}z^{-2}z^{3}z^{3}z^{0}z^{3}z^{2}z^{-2}z^{5}z^{7}z^{-7}z^{0}z^{0}=1,
\end{aligned}$$

$$\begin{aligned}
&[h,x,y,a,[b,y],c,x][h,a,x,y,[b,y],c,x][h,x,y,[b,y],a,c,x][h,a,[b,y],x,y,c,x]\\
&[h,[b,y],x,y,a,c,x][h,[b,y],a,x,y,c,x][h,a,x,y,[b,y],c,x]^{-1}[h,a,y,x,[b,y],c,x]^{-1}\\
&[h,x,a,[b,y],y,c,x][h,y,a,[b,y],x,c,x][h,x,[b,y],y,a,c,x][h,y,[b,y],x,a,c,x]\\
&[h,[b,y],a,x,y,c,x]^{-1}[h,[b,y],a,y,x,c,x]^{-1}\\
=&z^{0}z^{2}z^{-4}z^{-3}z^{2}z^{-1}z^{-2}z^{-3}z^{6}z^{-5}z^{-1}z^{3}zz^{5}=1.
\end{aligned}$$

Finally for $u = a$ and $v = [b, y]$ suppose that $t = x[x, y]$. Commuting Q by c gives

$$\begin{aligned}
&[h,x,[x,y],a,[b,y],c][h,a,x,[x,y],[b,y],c][h,x,[x,y],[b,y],a,c][h,a,[b,y],x,[x,y],c]\\
&[h,[b,y],x,[x,y],a,c][h,[b,y],a,x,[x,y],c][h,a,x,[x,y],[b,y],c]^{-1}\\
&[h,a,[x,y],x,[b,y],c]^{-1}[h,x,a,[b,y],[x,y],c][h,[x,y],a,[b,y],x,c][h,x,[b,y],[x,y],a,c]\\
&[h,[x,y],[b,y],x,a,c][h,[b,y],a,x,[x,y],c]^{-1}[h,[b,y],a,[x,y],x,c]^{-1}\\
=&z^{0}z^{7}z^{-7}z^{-2}z^{2}z^{0}z^{-7}z^{2}z^{-1}z^{-4}z^{5}z^{5}z^{0}z^{0}=1.
\end{aligned}$$

Next suppose that $u = [a, x]$ and $v = b$. First suppose that $t = x$. Note that swapping u and v in the weight 4 commutators in Q leaves Q unchanged. Hence this case follows from the case $u = a$, $v = [b, x]$ and $t = x$. The case $t = [x, y]$ follows in the same way. Consider $t = xy$. In this case

$$\begin{aligned}
Q=&[h,x,y,[a,x],b][h,[a,x],x,y,b][h,x,y,b,[a,x]][h,[a,x],b,x,y][h,b,x,y,[a,x]]\\
&[h,b,[a,x],x,y][h,[a,x],x,y,b]^{-1}[h,[a,x],y,x,b]^{-1}[h,x,[a,x],b,y][h,y,[a,x],b,x]\\
&[h,x,b,y,[a,x]][h,y,b,x,[a,x]][h,b,[a,x],x,y]^{-1}[h,b,[a,x],y,x]^{-1}\\
&[h,[a,x],x,y,y,b]^{-1}[h,[a,x],y,x,y,b]^{-1}[h,x,y,[a,x],b,y][h,y,[a,x],b,x,y]\\
&[h,x,y,b,y,[a,x]][h,y,b,x,y,[a,x]][h,b,[a,x],x,y,y]^{-1}[h,b,[a,x],y,x,y]^{-1}.
\end{aligned}$$

Commuting by c gives, in terms of (A.1),

$$
\begin{aligned}
&[h,x,y,y,[a,x],b,c][h,[a,x],x,y,y,b,c][h,x,y,y,b,[a,x],c][h,[a,x],b,x,y,y,c]\\
&[h,b,x,y,y,[a,x],c][h,b,[a,x],x,y,y,c][h,[a,x],x,y,y,b,c]^{-1}[h,[a,x],y,y,x,b,c]^{-1}\\
&[h,x,[a,x],b,y,y,c][h,y,y,[a,x],b,x,c][h,x,b,y,y,[a,x],c][h,y,y,b,x,[a,x],c]\\
&[h,b,[a,x],x,y,y,c]^{-1}[h,b,[a,x],y,y,x,c]^{-1}[h,[a,x],x,y,y,b,c]^{-2}\\
&[h,[a,x],y,x,y,b,c]^{-2}[h,x,y,[a,x],b,y,c]^{2}[h,y,[a,x],b,x,y,c]^{2}[h,x,y,b,y,[a,x],c]^{2}\\
&[h,y,b,x,y,[a,x],c]^{2}[h,b,[a,x],x,y,y,c]^{-2}[h,b,[a,x],y,x,y,c]^{-2}\\
&=z^{-1}z^{-4}z^{0}z^{0}zz^{4}z^{4}z^{-1}z^{-6}z^{-5}z^{5}z^{6}z^{-4}zz^{8}z^{-6}z^{6}z^{10}z^{-10}z^{-6}z^{-8}z^{6}=1.
\end{aligned}
$$

Commuting instead by y,c or by c,y gives

$$
\begin{aligned}
&[h,x,y,[a,x],b,y,c][h,[a,x],x,y,b,y,c][h,x,y,b,[a,x],y,c][h,[a,x],b,x,y,y,c]\\
&[h,b,x,y,[a,x],y,c][h,b,[a,x],x,y,y,c][h,[a,x],x,y,b,y,c]^{-1}[h,[a,x],y,x,b,y,c]^{-1}\\
&[h,x,[a,x],b,y,y,c][h,y,[a,x],b,x,y,c][h,x,b,y,[a,x],y,c][h,y,b,x,[a,x],y,c]\\
&[h,b,[a,x],x,y,y,c]^{-1}[h,b,[a,x],y,x,y,c]^{-1}\\
&=z^{3}z^{-2}zz^{0}z^{-2}z^{4}z^{2}z^{3}z^{-6}z^{5}z^{0}z^{-7}z^{-4}z^{3}=1,
\end{aligned}
$$

$$
\begin{aligned}
&[h,x,y,[a,x],b,c,y][h,[a,x],x,y,b,c,y][h,x,y,b,[a,x],c,y][h,[a,x],b,x,y,c,y]\\
&[h,b,x,y,[a,x],c,y][h,b,[a,x],x,y,c,y][h,[a,x],x,y,b,c,y]^{-1}[h,[a,x],y,x,b,c,y]^{-1}\\
&[h,x,[a,x],b,y,c,y][h,y,[a,x],b,x,c,y][h,x,b,y,[a,x],c,y][h,y,b,x,[a,x],c,y]\\
&[h,b,[a,x],x,y,c,y]^{-1}[h,b,[a,x],y,x,c,y]^{-1}\\
&=z^{-3}z^{3}z^{0}z^{5}z^{-3}z^{2}z^{-3}z^{2}z^{2}z^{-5}z^{-1}z^{6}z^{-2}z^{-3}=1.
\end{aligned}
$$

Now consider $u=[a,y]$, $v=b$ and $t=x$. Here

$$
\begin{aligned}
Q=&[h,x,[a,y],b][h,[a,y],x,b][h,x,b,[a,y]][h,[a,y],b,x][h,b,x,[a,y]][h,b,[a,y],x]\\
&[h,[a,y],x,x,b]^{-1}[h,x,[a,y],b,x][h,x,b,x,[a,y]][h,b,[a,y],x,x]^{-1}.
\end{aligned}
$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$
\begin{aligned}
&([h,x,x,y,a,y,b,c][h,y,a,y,x,x,b,c][h,x,x,b,y,a,y,c][h,y,a,y,b,x,x,c]\\
&\ [h,b,x,x,y,a,y,c][h,b,y,a,y,x,x,c][h,y,a,y,x,x,b,c]^{-2}[h,x,y,a,y,b,x,c]^{2}\\
&\ [h,x,b,x,y,a,y,c]^{2}[h,b,y,a,y,x,x,c]^{-2})^{-3}=(zzz^{4}z^{-4}z^{-1}z^{-1}z^{-2}z^{6}z^{-6}z^{2})^{-3}=1.
\end{aligned}
$$

Commuting instead by y, c or c, y gives

$$[h,x,x,[a,y],b,y,c][h,[a,y],x,x,b,y,c][h,x,x,b,[a,y],y,c][h,[a,y],b,x,x,y,c]$$
$$[h,b,x,x,[a,y],y,c][h,b,[a,y],x,x,y,c][h,[a,y],x,x,b,y,c]^{-2}[h,x,[a,y],b,x,y,c]^2$$
$$[h,x,b,x,[a,y],y,c]^2[h,b,[a,y],x,x,y,c]^{-2} = z^{-5}z^5z^{-6}z^0z^{-1}z^{-1}z^{-10}z^2z^{14}z^2 = 1,$$

$$[h,x,x,[a,y],b,c,y][h,[a,y],x,x,b,c,y][h,x,x,b,[a,y],c,y][h,[a,y],b,x,x,c,y]$$
$$[h,b,x,x,[a,y],c,y][h,b,[a,y],x,x,c,y][h,[a,y],x,x,b,c,y]^{-2}[h,x,[a,y],b,x,c,y]^2$$
$$[h,x,b,x,[a,y],c,y]^2[h,b,[a,y],x,x,c,y]^{-2} = z^{-1}z^{-1}z^0z^{-6}z^5z^{-5}z^2z^6z^{-10}z^{10} = 1.$$

Commuting by x, c, by c, x or by three elements follows from the case $u = a$, $v = [b, y]$ and $t = x$ by swapping u and v, as only the smallest weight terms in P are involved. Next suppose that $t = xy$. Then

$$\begin{aligned}Q =&[h,x,y,[a,y],b][h,[a,y],x,y,b][h,x,y,b,[a,y]][h,[a,y],b,x,y][h,b,x,y,[a,y]]\\&[h,b,[a,y],x,y][h,[a,y],x,y,b]^{-1}[h,[a,y],y,x,b]^{-1}[h,x,[a,y],b,y][h,y,[a,y],b,x]\\&[h,x,b,y,[a,y]][h,y,b,x,[a,y]][h,b,[a,y],x,y]^{-1}[h,b,[a,y],y,x]^{-1}\\&[h,[a,y],x,x,y,b]^{-1}[h,[a,y],x,y,x,b]^{-1}[h,x,[a,y],b,x,y][h,x,y,[a,y],b,x]\\&[h,x,b,x,y,[a,y]][h,x,y,b,x,[a,y]][h,b,[a,y],x,x,y]^{-1}[h,b,[a,y],x,y,x]^{-1}.\end{aligned}$$

Commuting on the right by c gives, in terms of (A.1),

$$\begin{aligned}&[h,x,x,y,[a,y],b,c][h,[a,y],x,x,y,b,c][h,x,x,y,b,[a,y],c][h,[a,y],b,x,x,y,c]\\&[h,b,x,x,y,[a,y],c][h,b,[a,y],x,x,y,c][h,[a,y],x,x,y,b,c]^{-1}[h,[a,y],y,x,x,b,c]^{-1}\\&[h,x,x,[a,y],b,y,c][h,y,[a,y],b,x,x,c][h,x,x,b,y,[a,y],c][h,y,b,x,x,[a,y],c]\\&[h,b,[a,y],x,x,y,c]^{-1}[h,b,[a,y],y,x,x,c]^{-1}[h,[a,y],x,x,y,b,c]^{-2}\\&[h,[a,y],x,y,x,b,c]^{-2}[h,x,[a,y],b,x,y,c]^2[h,x,y,[a,y],b,x,c]^2[h,x,b,x,y,[a,y],c]^2\\&[h,x,y,b,x,[a,y],c]^2[h,b,[a,y],x,x,y,c]^{-2}[h,b,[a,y],x,y,x,c]^{-2}\\=&z^4zz^0z^0z^{-4}z^{-1}z^{-1}z^4z^{-5}z^{-6}z^6z^5zz^{-4}z^{-2}z^{-6}z^2z^4z^{-4}z^{-2}z^2z^6 = 1.\end{aligned}$$

Commuting instead by x, c or by c, x gives

$$\begin{aligned}
&[h,x,y,[a,y],b,x,c][h,[a,y],x,y,b,x,c][h,x,y,b,[a,y],x,c][h,[a,y],b,x,y,x,c]\\
&[h,b,x,y,[a,y],x,c][h,b,[a,y],x,y,x,c][h,[a,y],x,y,b,x,c]^{-1}[h,[a,y],y,x,b,x,c]^{-1}\\
&[h,x,[a,y],b,y,x,c][h,y,[a,y],b,x,x,c][h,x,b,y,[a,y],x,c][h,y,b,x,[a,y],x,c]\\
&[h,b,[a,y],x,y,x,c]^{-1}[h,b,[a,y],y,x,x,c]^{-1}\\
=&z^2z^{-3}z^5z^0z^3z^{-3}z^3z^2z^5z^{-6}z^{-7}z^0z^3z^{-4}=1,
\end{aligned}$$

$$\begin{aligned}
&[h,x,y,[a,y],b,c,x][h,[a,y],x,y,b,c,x][h,x,y,b,[a,y],c,x][h,[a,y],b,x,y,c,x]\\
&[h,b,x,y,[a,y],c,x][h,b,[a,y],x,y,c,x][h,[a,y],x,y,b,c,x]^{-1}[h,[a,y],y,x,b,c,x]^{-1}\\
&[h,x,[a,y],b,y,c,x][h,y,[a,y],b,x,c,x][h,x,b,y,[a,y],c,x][h,y,b,x,[a,y],c,x]\\
&[h,b,[a,y],x,y,c,x]^{-1}[h,b,[a,y],y,x,c,x]^{-1}\\
=&z^4z^{-2}z^0zz^{-2}z^3z^2z^{-3}z^{-5}z^2z^6z^{-1}z^{-3}z^{-2}=1.
\end{aligned}$$

Finally for this u and v consider $t = x[x, y]$. Commuting Q by c gives

$$\begin{aligned}
Q=&[h,x,[x,y],[a,y],b,c][h,[a,y],x,[x,y],b,c][h,x,[x,y],b,[a,y],c][h,[a,y],b,x,[x,y],c]\\
&[h,b,x,[x,y],[a,y],c][h,b,[a,y],x,[x,y],c][h,[a,y],x,[x,y],b,c]^{-1}\\
&[h,[a,y],[x,y],x,b,c]^{-1}[h,x,[a,y],b,[x,y],c][h,[x,y],[a,y],b,x,c]\\
&[h,x,b,[x,y],[a,y],c][h,[x,y],b,x,[a,y],c][h,b,[a,y],x,[x,y],c]^{-1}\\
&[h,b,[a,y],[x,y],x,c]^{-1}=z^7z^{-2}z^0z^0z^{-7}z^2z^2z^{-7}z^{-4}z^{-1}zz^4z^{-2}z^7=1.
\end{aligned}$$

Now consider $u = [a, x]$ and $v = [b, y]$. First suppose that $t = x$. Then

$$\begin{aligned}
Q=&[h,x,[a,x],[b,y]][h,[a,x],x,[b,y]][h,x,[b,y],[a,x]][h,[a,x],[b,y],x]\\
&[h,[b,y],x,[a,x]][h,[b,y],[a,x],x].
\end{aligned}$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$\begin{aligned}
&([h,x,[a,x],y,b,y,c][h,[a,x],x,y,b,y,c][h,x,y,b,y,[a,x],c][h,[a,x],y,b,y,x,c]\\
&[h,y,b,y,x,[a,x],c][h,y,b,y,[a,x],x,c])^{-3}=(z^7z^{-2}z^{-5}z^{-5}z^{-2}z^7)^{-3}=1.
\end{aligned}$$

Instead commuting by y, c gives

$$\begin{aligned}
&[h,x,[a,x],[b,y],y,c][h,[a,x],x,[b,y],y,c][h,x,[b,y],[a,x],y,c][h,[a,x],[b,y],x,y,c]\\
&[h,[b,y],x,[a,x],y,c][h,[b,y],[a,x],x,y,c]=z^{-13}z^8z^{-1}zz^{10}z^{-5}=1.
\end{aligned}$$

Commuting by c, y follows from this by cycling twice to the right and reflecting. Next suppose that $t = [x, y]$. Then commuting Q by c gives

$$\begin{aligned}&[h,[x,y],[a,x],[b,y],c][h,[a,x],[x,y],[b,y],c][h,[x,y],[b,y],[a,x],c]\\&[h,[a,x],[b,y],[x,y],c][h,[b,y],[x,y],[a,x],c][h,[b,y],[a,x],[x,y],c]\\&= z^{-4}z^{8}z^{-4}z^{-4}z^{8}z^{-4} = 1.\end{aligned}$$

Now suppose that $t = xy$. Then

$$\begin{aligned}Q =&[h,x,y,[a,x],[b,y]][h,[a,x],x,y,[b,y]][h,x,y,[b,y],[a,x]][h,[a,x],[b,y],x,y]\\&[h,[b,y],x,y,[a,x]][h,[b,y],[a,x],x,y][h,[a,x],x,y,[b,y]]^{-1}[h,[a,x],y,x,[b,y]]^{-1}\\&[h,x,[a,x],[b,y],y][h,y,[a,x],[b,y],x][h,x,[b,y],y,[a,x]][h,y,[b,y],x,[a,x]]\\&[h,[b,y],[a,x],x,y]^{-1}[h,[b,y],[a,x],y,x]^{-1}.\end{aligned}$$

Commuting by c gives $zz^2z^{-5}zz^6z^{-5}z^{-2}z^6z^{-13}zz^{10}z^{-8}z^5z = 1$.

For $u = [a, y]$, $v = [b, x]$ and $t = x$ we swap a and b in the case $u = [a, x]$, $v = [b, y]$ and $t = x$. Similarly for $t = [x, y]$. Consider $t = xy$. Then commuting Q by c gives

$$\begin{aligned}&[h,x,y,[a,y],[b,x],c][h,[a,y],x,y,[b,x],c][h,x,y,[b,x],[a,y],c][h,[a,y],[b,x],x,y,c]\\&[h,[b,x],x,y,[a,y],c][h,[b,x],[a,y],x,y,c][h,[a,y],x,y,[b,x],c]^{-1}\\&[h,[a,y],y,x,[b,x],c]^{-1}[h,x,[a,y],[b,x],y,c][h,y,[a,y],[b,x],x,c]\\&[h,x,[b,x],y,[a,y],c][h,y,[b,x],x,[a,y],c][h,[b,x],[a,y],x,y,c]^{-1}\\&[h,[b,x],[a,y],y,x,c]^{-1} = z^5z^{-6}z^{-1}z^5z^{-2}z^{-1}z^6z^{-2}zz^{-13}z^{-8}z^{10}zz^5 = 1.\end{aligned}$$

Now suppose that $u = a$ and $v = [b, x, y]$. Suppose that $t = x$. Then

$$\begin{aligned}Q =&[h,x,a,[b,x,y]][h,a,x,[b,x,y]][h,x,[b,x,y],a][h,a,[b,x,y],x]\\&[h,[b,x,y],x,a][h,[b,x,y],a,x].\end{aligned}$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$\begin{aligned}&[h,x,a,y,x,b,y,c]^3[h,x,a,y,b,x,y,c]^{-3}[h,a,x,y,x,b,y,c]^3[h,a,x,y,b,x,y,c]^{-3}\\&[h,x,y,x,b,y,a,c]^3[h,x,y,b,x,y,a,c]^{-3}[h,a,y,x,b,y,x,c]^3[h,a,y,b,x,y,x,c]^{-3}\\&[h,y,x,b,y,x,a,c]^3[h,y,b,x,y,x,a,c]^{-3}[h,y,x,b,y,a,x,c]^3[h,y,b,x,y,a,x,c]^{-3}\\=&z^3z^{-3}z^{-6}z^0z^6z^0z^0z^6z^0z^{-6}z^{-3}z^3 = 1.\end{aligned}$$

Commuting instead by y, c gives

$$[h, x, a, [b, x, y], y, c][h, a, x, [b, x, y], y, c][h, x, [b, x, y], a, y, c][h, a, [b, x, y], x, y, c]$$
$$[h, [b, x, y], x, a, y, c][h, [b, x, y], a, x, y, c] = z^5 z^{-3} z^{-4} z^0 z^{-4} z^6 = 1.$$

Commuting by c, y follows by cycling to the right twice and reflecting, which, by Lemma A.1, gives the same as the previous case. Next suppose that $t = [x, y]$. Then

$$[h, [x, y], a, [b, x, y], c][h, a, [x, y], [b, x, y], c][h, [x, y], [b, x, y], a, c][h, a, [b, x, y], [x, y], c]$$
$$[h, [b, x, y], [x, y], a, c][h, [b, x, y], a, [x, y], c] = z^0 z^3 z^{-3} z^{-3} z^3 z^0 = 1.$$

Now suppose that $t = xy$. Then commuting Q by c gives

$$\begin{aligned}
&[h, x, y, a, [b, x, y], c][h, a, x, y, [b, x, y], c][h, x, y, [b, x, y], a, c][h, a, [b, x, y], x, y, c] \\
&[h, [b, x, y], x, y, a, c][h, [b, x, y], a, x, y, c][h, a, x, y, [b, x, y], c]^{-1}[h, a, y, x, [b, x, y], c]^{-1} \\
&[h, x, a, [b, x, y], y, c][h, y, a, [b, x, y], x, c][h, x, [b, x, y], y, a, c][h, y, [b, x, y], x, a, c] \\
&[h, [b, x, y], a, x, y, c]^{-1}[h, [b, x, y], a, y, x, c]^{-1} \\
=&z^{-6} z^3 z^{-3} z^0 z^0 z^6 z^{-3} z^0 z^5 z^4 z^3 z^3 z^{-6} z^{-6} = 1.
\end{aligned}$$

Now consider the case $u = a$ and $v = [b, y, x]$. First consider $t = x$. Then

$$\begin{aligned}
Q =&[h, x, a, [b, y, x]][h, a, x, [b, y, x]][h, x, [b, y, x], a][h, a, [b, y, x], x] \\
&[h, [b, y, x], x, a][h, [b, y, x], a, x].
\end{aligned}$$

Commuting by c and using Section A.3 gives, in terms of (A.1),

$$\begin{aligned}
&[h, x, a, x, y, b, y, c]^3[h, x, a, y, b, y, x, c]^{-3}[h, a, x, x, y, b, y, c]^3[h, a, x, y, b, y, x, c]^{-3} \\
&[h, x, x, y, b, y, a, c]^3[h, x, y, b, y, x, a, c]^{-3}[h, a, x, y, b, y, x, c]^3[h, a, y, b, y, x, x, c]^{-3} \\
&[h, x, y, b, y, x, a, c]^3[h, y, b, y, x, x, a, c]^{-3}[h, x, y, b, y, a, x, c]^3[h, y, b, y, x, a, x, c]^{-3} \\
=&z^9 z^{-9} z^3 z^6 z^{-3} z^{-6} z^{-6} z^{-3} z^6 z^3 z^{-9} z^9 = 1.
\end{aligned}$$

Commuting instead by y, c gives

$$[h, x, a, [b, y, x], y, c][h, a, x, [b, y, x], y, c][h, x, [b, y, x], a, y, c][h, a, [b, y, x], x, y, c]$$
$$[h, [b, y, x], x, a, y, c][h, [b, y, x], a, x, y, c] = z^4 z^{-3} z^{-5} z^3 z^{-5} z^6 = 1.$$

Commuting by c, y follows by cycling to the right twice and reflecting, which, by Lemma

A.1, gives the same as the previous case. The case $t = [x, y]$ follows from the case $u = a$, $v = [b, x, y]$ and $t = [x, y]$ by swapping x and y, since $[x, y] = [y, x]^{-1}$. Now consider $t = xy$. Then commuting Q by c gives

$$\begin{aligned}
&[h, x, y, a, [b, y, x], c][h, a, x, y, [b, y, x], c][h, x, y, [b, y, x], a, c][h, a, [b, y, x], x, y, c]\\
&[h, [b, y, x], x, y, a, c][h, [b, y, x], a, x, y, c][h, a, x, y, [b, y, x], c]^{-1}[h, a, y, x, [b, y, x], c]^{-1}\\
&[h, x, a, [b, y, x], y, c][h, y, a, [b, y, x], x, c][h, x, [b, y, x], y, a, c][h, y, [b, y, x], x, a, c]\\
&[h, [b, y, x], a, x, y, c]^{-1}[h, [b, y, x], a, y, x, c]^{-1}\\
=&z^{-6}z^0z^0z^3z^{-3}z^6z^0z^{-3}z^4z^5z^3z^3z^{-6}z^{-6} = 1.
\end{aligned}$$

Now suppose that $u = [a, x, y]$ and $v = b$. Since we can swap u and v in the lowest weight terms in Q we are done for $t = x$ and $t = [x, y]$. It remains to consider $t = xy$. Here commuting Q by c gives

$$\begin{aligned}
&[h, x, y, [a, x, y], b, c][h, [a, x, y], x, y, b, c][h, x, y, b, [a, x, y], c][h, [a, x, y], b, x, y, c]\\
&[h, b, x, y, [a, x, y], c][h, b, [a, x, y], x, y, c][h, [a, x, y], x, y, b, c]^{-1}[h, [a, x, y], y, x, b, c]^{-1}\\
&[h, x, [a, x, y], b, y, c][h, y, [a, x, y], b, x, c][h, x, b, y, [a, x, y], c][h, y, b, x, [a, x, y], c]\\
&[h, b, [a, x, y], x, y, c]^{-1}[h, b, [a, x, y], y, x, c]^{-1}\\
=&z^3z^0z^6z^{-6}z^{-3}z^0z^0z^{-3}z^4z^5z^{-5}z^{-4}z^0z^3 = 1.
\end{aligned}$$

Similarly for $u = [a, y, x]$ and $v = b$ we need just consider $t = xy$. Then commuting Q by c gives

$$\begin{aligned}
&[h, x, y, [a, y, x], b, c][h, [a, y, x], x, y, b, c][h, x, y, b, [a, y, x], c][h, [a, y, x], b, x, y, c]\\
&[h, b, x, y, [a, y, x], c][h, b, [a, y, x], x, y, c][h, [a, y, x], x, y, b, c]^{-1}[h, [a, y, x], y, x, b, c]^{-1}\\
&[h, x, [a, y, x], b, y, c][h, y, [a, y, x], b, x, c][h, x, b, y, [a, y, x], c][h, y, b, x, [a, y, x], c]\\
&[h, b, [a, y, x], x, y, c]^{-1}[h, b, [a, y, x], y, x, c]^{-1}\\
=&z^0z^3z^6z^{-6}z^0z^{-3}z^{-3}z^0z^5z^4z^{-4}z^{-5}z^3z^0 = 1.
\end{aligned}$$

Next suppose that $u = [a, x]$ and $v = [b, x, y]$ or $[b, y, x]$. Since commutators with a three x entries are trivial these cases hold. Similarly for $u = [a, x, y]$ or $[a, y, x]$ and $v = [b, y]$. Suppose that $u = [a, y]$, $v = [b, x, y]$ and $t = x$. Then commuting Q by c gives

$$\begin{aligned}
&[h, x, [a, y], [b, x, y], c][h, [a, y], x, [b, x, y], c][h, x, [b, x, y], [a, y], c][h, [a, y], [b, x, y], x, c]\\
&[h, [b, x, y], x, [a, y], c][h, [b, x, y], [a, y], x, c] = z^{11}z^{-4}z^{-7}z^{-7}z^{-4}z^{11} = 1.
\end{aligned}$$

If $u = [a, y]$, $v = [b, y, x]$ and $t = x$, then commuting Q by c gives

$$[h, x, [a, y], [b, y, x], c][h, [a, y], x, [b, y, x], c][h, x, [b, y, x], [a, y], c][h, [a, y], [b, y, x], x, c]$$
$$[h, [b, y, x], x, [a, y], c][h, [b, y, x], [a, y], x, c] = z^{10}z^{-2}z^{-8}z^{-8}z^{-2}z^{10} = 1.$$

The cases $u = [a, x, y]$, $v = [b, y]$, $t = x$ and $u = [a, y, x]$, $v = [b, y]$, $t = x$ follow by swapping u and v in Q.

Now suppose that $u = a$ and $v = [b, x, y, x]$ or $u = [a, x, y, x]$ and $v = b$. These cases are trivial as every commutator will contain at least three x entries. Suppose that $u = a$ and $v = [b, y, x, y]$. The only case to check is $t = x$. Here commuting Q by c gives

$$[h, x, a, [b, y, x, y], c][h, a, x, [b, y, x, y], c][h, x, [b, y, x, y], a, c][h, a, [b, y, x, y], x, c]$$
$$[h, [b, y, x, y], x, a, c][h, [b, y, x, y], a, x, c] = z^0z^{-3}z^3z^3z^{-3}z^0 = 1.$$

The case $u = [a, y, x, y]$ and $v = b$ follows by swapping u and v in Q. This completes the proof of (iii), since if u and v were of higher combined weight, then there would be at least three x entries in the product for g.

Proof of (iv): Let

$$R = [h, u, v, w][h, v, u, w][h, u, w, v][h, v, w, u][h, w, u, v][h, w, v, u].$$

If any two of u, v and w are of the same form, then by the alternating property we have that $R = 1$. Hence we assume that u, v and w are all of different forms. If the sum of the weights of u, v and w is 7, then we are done by Lemma A.1 and reflecting. If all are weight 2, then two must be of the same form. So we may assume that one is of weight 1. By the symmetry of R we may therefore assume, without loss of generality, that $u = a$. Neither v or w are weight 1 and so at least one is weight 2. Again without loss of generality we may assume that v is. As t doesn't appear in R and w is not of weight 4 the problem is symmetric in x and y and thus we may assume further that $v = [b, x]$. It remains to consider $w = [c, y]$, $w = [c, x, y]$ and $w = [c, y, x]$.

First consider $w = [c, y]$. Here

$$\begin{aligned} R =& [h, a, [b, x], [c, y]][h, [b, x], a, [c, y]][h, a, [c, y], [b, x]] \\ & [h, [b, x], [c, y], a][h, [c, y], a, [b, x]][h, [c, y], [b, x], a]. \end{aligned}$$

Using section A.3 this gives, in terms of (A.1),

$$([h,a,x,b,x,y,c,y][h,x,b,x,a,y,c,y][h,a,y,c,y,x,b,x][h,x,b,x,y,c,y,a]$$
$$[h,y,c,y,a,x,b,x][h,y,c,y,x,b,x,a])^9 = (z^3 z^{-3} z^{-3} z^3 z^3 z^{-3})^9 = 1.$$

Commuting by x before using Section A.3 gives

$$([h,a,[b,x],y,c,y,x][h,[b,x],a,y,c,y,x][h,a,y,c,y,[b,x],x][h,[b,x],y,c,y,a,x]$$
$$[h,y,c,y,a,[b,x],x][h,y,c,y,[b,x],a,x])^{-3} = (z^{-5} z^{-2} z^7 z^7 z^{-2} z^{-5})^{-3} = 1.$$

Commuting instead by y follows from this by swapping x and y. Commuting instead by x,y gives

$$[h,a,[b,x],[c,y],x,y][h,[b,x],a,[c,y],x,y][h,a,[c,y],[b,x],x,y][h,[b,x],[c,y],a,x,y]$$
$$[h,[c,y],a,[b,x],x,y][h,[c,y],[b,x],a,x,y] = zz^6 z^{-5} zz^2 z^{-5} = 1.$$

Commuting by y,x follows by swapping x and y.

Now suppose that $w = [c,x,y]$. Then

$$R = [h,a,[b,x],[c,x,y]][h,[b,x],a,[c,x,y]][h,a,[c,x,y],[b,x]]$$
$$[h,[b,x],[c,x,y],a][h,[c,x,y],a,[b,x]][h,[c,x,y],[b,x],a].$$

Using Section A.3 this gives, in terms of (A.1),

$$([h,a,[b,x],y,x,c,y][h,[b,x],a,y,c,x,y]^{-1}[h,a,y,x,c,y,[b,x]]$$
$$[h,[b,x],y,c,x,y,a]^{-1}[h,y,x,c,y,a,[b,x]][h,y,c,x,y,[b,x],a]^{-1})^3,$$

which is trivial by Lemma A.1 and reflecting. Instead commuting by y gives

$$[h,a,[b,x],[c,x,y],y][h,[b,x],a,[c,x,y],y][h,a,[c,x,y],[b,x],y][h,[b,x],[c,x,y],a,y]$$
$$[h,[c,x,y],a,[b,x],y][h,[c,x,y],[b,x],a,y] = z^{-8} z^{-2} z^{10} z^{10} z^{-2} z^{-8} = 1.$$

Finally suppose that $w = [c,y,x]$. Then

$$R = [h,a,[b,x],[c,y,x]][h,[b,x],a,[c,y,x]][h,a,[c,y,x],[b,x]]$$
$$[h,[b,x],[c,y,x],a][h,[c,y,x],a,[b,x]][h,[c,y,x],[b,x],a].$$

Using Section A.3 this again satisfies (A.1) by Lemma A.1 and reflecting. Instead commuting by y gives

$$[h,a,[b,x],[c,y,x],y][h,[b,x],a,[c,y,x],y][h,a,[c,y,x],[b,x],y][h,[b,x],[c,y,x],a,y]$$
$$[h,[c,y,x],a,[b,x],y][h,[c,y,x],[b,x],a,y]=z^{-7}z^{-4}z^{11}z^{11}z^{-4}z^{-7}=1.$$

This finishes the proof of (iv).

<u>Proof of (v):</u> Let

$$\begin{aligned}S=&[h,v,w,t,u][h,w,t,u,v][h,u,v,w,t][h,u,w,t,v][h,t,w,v,u]^{-1}[h,t,v,u,w]^{-1}\\&[h,v,u,t,w]^{-1}[h,w,v,u,t]^{-1}[h,t,v,w,t,u][h,t,u,v,w,t][h,t,w,t,u,v]\\&[h,t,u,w,t,v][h,v,u,t,t,w][h,w,v,u,t,t].\end{aligned}$$

Note that if the sum of weights of u, v and w is six, then, by cycling,

$$\begin{aligned}S=&[h,u,v,w,t][h,u,v,w,t][h,u,v,w,t][h,v,u,w,t][h,t,w,v,u]^{-1}[h,v,u,w,t]^{-1}\\&[h,t,w,v,u]^{-1}[h,t,w,v,u]^{-1}=[h,u,v,w,t]^{3}[h,t,w,v,u]^{-3}.\end{aligned}\tag{A.2}$$

By Lemma A.1 and reflecting this is trivial. Hence we only need consider cases where the sum of weights of u, v and w is less than six. Also, when $t=[x,y]$ and the sum of weights of u, v and w is five, by the same reason S is trivial. We can also use (A.2) for the weight five part of S when we are not commuting by any elements.

First suppose that $u=a$, $v=b$ and $w=c$. Then, by the alternating property,

$$\begin{aligned}S=&[h,a,b,t,c][h,a,t,b,c][h,a,b,c,t][h,a,b,t,c]^{-1}[h,t,a,b,c][h,t,a,b,c]\\&[h,a,b,t,c][h,a,b,c,t][h,t,a,b,t,c][h,t,a,b,c,t][h,t,a,t,b,c]\\&[h,t,a,b,t,c]^{-1}[h,a,b,t,t,c]^{-1}[h,a,b,c,t,t]^{-1}\\=&[h,a,t,b,c][h,a,b,c,t]^{2}[h,t,a,b,c]^{2}[h,a,b,t,c]\\&[h,t,a,b,c,t][h,t,a,t,b,c][h,a,b,t,t,c]^{-1}[h,a,b,c,t,t]^{-1}.\end{aligned}$$

Suppose that $t=x$. Commuting by y or y,y gives, in terms of (A.1),

$$\begin{aligned}&[h,a,x,x,b,c,y,y][h,a,b,c,x,x,y,y]^{2}[h,x,x,a,b,c,y,y]^{2}[h,a,b,x,x,c,y,y]\\&[h,x,a,b,c,x,y,y]^{2}[h,x,a,x,b,c,y,y]^{2}[h,a,b,x,x,c,y,y]^{-2}[h,a,b,c,x,x,y,y]^{-2}\\=&z^{2}z^{-6}z^{-6}z^{2}z^{4}z^{2}z^{-4}z^{6}=1.\end{aligned}$$

Commuting instead by x, y or x, y, y gives

$$[h, a, x, b, c, x, y, y][h, a, b, c, x, x, y, y]^2[h, x, a, b, c, x, y, y]^2[h, a, b, x, c, x, y, y]$$
$$= zz^{-6}z^4z = 1$$

and commuting by y, x or y, y, x gives

$$[h, a, x, b, c, y, y, x][h, a, b, c, x, y, y, x]^2[h, x, a, b, c, y, y, x]^2[h, a, b, x, c, y, y, x]$$
$$= zz^4z^{-6}z = 1.$$

Commuting instead by y, x, y gives

$$[h, a, x, b, c, y, x, y][h, a, b, c, x, y, x, y]^2[h, x, a, b, c, y, x, y]^2[h, a, b, x, c, y, x, y]$$
$$= z^{-2}z^2z^2z^{-2} = 1.$$

Now suppose that $t = [x, y]$. Then

$$S = [h, a, [x, y], b, c][h, a, b, c, [x, y]]^2[h, [x, y], a, b, c]^2[h, a, b, [x, y], c]z^6.$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$[h, a, b, c, x, y, y, x]^{-12}[h, a, b, c, x, y, x, y]^{12}[h, x, y, y, x, c, b, a]^{12}$$
$$[h, x, y, x, y, a, b, c]^{-12}z^{24} = z^{-24}z^{12}z^{-24}z^{12}z^{24} = 1.$$

Instead commuting by x first gives

$$([h, a, y, x, y, b, c, x][h, a, b, c, y, x, y, x]^2[h, y, x, y, a, b, c, x]^2[h, a, b, y, x, y, c, x])^{-3}$$
$$= (z^{-2}z^2z^2z^{-2})^{-3} = 1.$$

Instead commuting by y first follows by swapping x and y. Commuting by x, y gives

$$[h, a, [x, y], b, c, x, y][h, a, b, c, [x, y], x, y]^2[h, [x, y], a, b, c, x, y]^2[h, a, b, [x, y], c, x, y]$$
$$=z^2z^{-2}z^{-2}z^2 = 1.$$

Commuting by y, x follows from this by swapping x and y. Next suppose that $t = xy$.

Then

$$
\begin{aligned}
S =&[h,a,x,y,b,c][h,a,b,c,x,y]^2[h,x,y,a,b,c]^2[h,a,b,x,y,c][h,x,a,b,c,y] \\
&[h,y,a,b,c,x][h,x,a,b,c,x,y][h,x,y,a,b,c,x][h,y,a,b,c,x,y][h,x,y,a,b,c,y] \\
&[h,x,y,a,b,c,x,y][h,x,a,y,b,c][h,y,a,x,b,c][h,x,a,x,y,b,c][h,x,y,a,x,b,c] \\
&[h,y,a,x,y,b,c][h,x,y,a,y,b,c][h,x,y,a,x,y,b,c][h,a,b,x,y,c]^{-1}[h,a,b,y,x,c]^{-1} \\
&[h,a,b,x,x,y,c]^{-1}[h,a,b,x,y,x,c]^{-1}[h,a,b,y,x,y,c]^{-1}[h,a,b,x,y,y,c]^{-1} \\
&[h,a,b,x,y,x,y,c]^{-1}[h,a,b,c,x,y]^{-1}[h,a,b,c,y,x]^{-1}[h,a,b,c,x,x,y]^{-1} \\
&[h,a,b,c,x,y,x]^{-1}[h,a,b,c,y,x,y]^{-1}[h,a,b,c,x,y,y]^{-1}[h,a,b,c,x,y,x,y]^{-1}.
\end{aligned}
$$

In terms of (A.1) this gives

$$
\begin{aligned}
&z^{-3}z^{-6}z^{-6}z^{-3}z^{-3}z^{-3}z^{4}z^{-6}z^{-6}z^{4}z^{4}z^{2}z^{2}z^{2}z^{2}z^{2}z^{2}z^{0}z^{3}z^{3}z^{6}z^{-4}z^{-4}z^{6}z^{-4}z^{3}z^{3}z^{6}z^{-4} \\
&z^{-4}z^{6}z^{-4} = 1.
\end{aligned}
$$

Instead commuting by x gives, in terms of (A.1),

$$
\begin{aligned}
&[h,a,x,y,y,b,c,x][h,a,b,c,x,y,y,x]^2[h,x,y,y,a,b,c,x]^2[h,a,b,x,y,y,c,x] \\
&[h,x,a,b,c,y,y,x][h,y,y,a,b,c,x,x][h,y,a,b,c,x,y,x]^2[h,x,y,a,b,c,y,x]^2 \\
&[h,x,a,y,y,b,c,x][h,y,y,a,x,b,c,x][h,y,a,x,y,b,c,x]^2[h,x,y,a,y,b,c,x]^2 \\
&[h,a,b,x,y,y,c,x]^{-1}[h,a,b,y,y,x,c,x]^{-1}[h,a,b,y,x,y,c,x]^{-2}[h,a,b,x,y,y,c,x]^{-2} \\
&[h,a,b,c,x,y,y,x]^{-1}[h,a,b,c,y,y,x,x]^{-1}[h,a,b,c,y,x,y,x]^{-2}[h,a,b,c,x,y,y,x]^{-2} \\
=&zz^{4}z^{-6}zz^{-3}z^{-3}z^{2}z^{4}z^{2}zz^{0}z^{2}z^{-1}z^{-1}z^{4}z^{-2}z^{-2}z^{3}z^{-2}z^{-4} = 1.
\end{aligned}
$$

Instead commuting by y gives

$$
\begin{aligned}
&[h,a,x,x,y,b,c,y][h,a,b,c,x,x,y,y]^2[h,x,x,y,a,b,c,y]^2[h,a,b,x,x,y,c,y] \\
&[h,x,x,a,b,c,y,y][h,y,a,b,c,x,x,y][h,x,a,b,c,x,y,y]^2[h,x,y,a,b,c,x,y]^2 \\
&[h,x,x,a,y,b,c,y][h,y,a,x,x,b,c,y][h,x,a,x,y,b,c,y]^2[h,x,y,a,x,b,c,y]^2 \\
&[h,a,b,x,x,y,c,y]^{-1}[h,a,b,y,x,x,c,y]^{-1}[h,a,b,x,x,y,c,y]^{-2}[h,a,b,x,y,x,c,y]^{-2} \\
&[h,a,b,c,x,x,y,y]^{-1}[h,a,b,c,y,x,x,y]^{-1}[h,a,b,c,x,x,y,y]^{-2}[h,a,b,c,x,y,x,y]^{-2} \\
=&zz^{-6}z^{4}zz^{-3}z^{-3}z^{4}z^{2}zz^{2}z^{-4}z^{-4}z^{-1}z^{-1}z^{-2}z^{4}z^{3}z^{-2}z^{6}z^{-2} = 1.
\end{aligned}
$$

Commuting by x, y or by y, x gives

$$\begin{aligned}
&[h,a,x,y,b,c,x,y][h,a,b,c,x,y,x,y]^2[h,x,y,a,b,c,x,y]^2[h,a,b,x,y,c,x,y]\\
&[h,x,a,b,c,y,x,y][h,y,a,b,c,x,x,y][h,x,a,y,b,c,x,y][h,y,a,x,b,c,x,y]\\
&[h,a,b,x,y,c,x,y]^{-1}[h,a,b,y,x,c,x,y]^{-1}[h,a,b,c,x,y,x,y]^{-1}[h,a,b,c,y,x,x,y]^{-1}\\
=&z^0z^2z^2z^0zz^{-3}z^{-2}zz^0z^2z^{-1}z^{-2}=1,
\end{aligned}$$

$$\begin{aligned}
&[h,a,x,y,b,c,y,x][h,a,b,c,x,y,y,x]^2[h,x,y,a,b,c,y,x]^2[h,a,b,x,y,c,y,x]\\
&[h,x,a,b,c,y,y,x][h,y,a,b,c,x,y,x][h,x,a,y,b,c,y,x][h,y,a,x,b,c,y,x]\\
&[h,a,b,x,y,c,y,x]^{-1}[h,a,b,y,x,c,y,x]^{-1}[h,a,b,c,x,y,y,x]^{-1}[h,a,b,c,y,x,y,x]^{-1}\\
=&z^{-2}z^4z^4z^{-2}z^{-3}zzz^{-2}z^2z^0z^{-2}z^{-1}=1.
\end{aligned}$$

Next suppose that $t = x[x, y]$. Then

$$\begin{aligned}
S=&[h,a,x,[x,y],b,c][h,a,b,c,x,[x,y]]^2[h,x,[x,y],a,b,c]^2[h,a,b,x,[x,y],c]\\
&[h,x,a,b,c,[x,y]][h,[x,y],a,b,c,x][h,x,a,[x,y],b,c][h,[x,y],a,x,b,c]\\
&[h,a,b,x,[x,y],c]^{-1}[h,a,b,[x,y],x,c]^{-1}[h,a,b,c,x,[x,y]]^{-1}[h,a,b,c,[x,y],x]^{-1}.
\end{aligned}$$

In terms of (A.1) this gives, using (A.2),

$$\begin{aligned}
&([h,a,b,c,x,y,x,y]^3[h,x,y,x,y,c,b,a]^{-3}[h,x,a,b,c,y,x,y][h,y,x,y,a,b,c,x]\\
&[h,x,a,y,x,y,b,c][h,y,x,y,a,x,b,c][h,a,b,x,y,x,y,c]^{-1}[h,a,b,y,x,y,x,c]^{-1}\\
&[h,a,b,c,x,y,x,y]^{-1}[h,a,b,c,y,x,y,x]^{-1})^{-3}=(z^3z^3zzz^{-2}z^{-2}z^{-1}z^{-1}z^{-1}z^{-1})^{-3}=1.
\end{aligned}$$

Instead commuting by y gives

$$\begin{aligned}
&[h,a,x,[x,y],b,c,y][h,a,b,c,x,[x,y],y]^2[h,x,[x,y],a,b,c,y]^2[h,a,b,x,[x,y],c,y]\\
&[h,x,a,b,c,[x,y],y][h,[x,y],a,b,c,x,y][h,x,a,[x,y],b,c,y][h,[x,y],a,x,b,c,y]\\
&[h,a,b,x,[x,y],c,y]^{-1}[h,a,b,[x,y],x,c,y]^{-1}[h,a,b,c,x,[x,y],y]^{-1}\\
&[h,a,b,c,[x,y],x,y]^{-1}=z^3z^{-8}z^2z^3zz^{-1}z^{-2}z^{-3}z^{-3}z^3z^4z=1.
\end{aligned}$$

Finally for $u = a$, $v = b$ and $w = c$ suppose that $t = xy[x, y]$. Then, using (A.2),

$$\begin{aligned}
S=&[h,a,b,c,x,y,[x,y]]^3[h,x,y,[x,y],c,b,a]^{-3}[h,x,y,a,b,c,[x,y]][h,[x,y],a,b,c,x,y]\\
&[h,x,y,a,[x,y],b,c][h,[x,y],a,x,y,b,c][h,a,b,x,y,[x,y],c]^{-1}[h,a,b,[x,y],x,y,c]^{-1}\\
&[h,a,b,c,x,y,[x,y]]^{-1}[h,a,b,c,[x,y],x,y]^{-1}=z^{-3}z^{-3}z^{-1}z^{-1}z^2z^2zzzz=1.
\end{aligned}$$

Now consider the case u and v are of the same form. So S is alternating in u and v and we get

$$\begin{aligned}S =&[h, w, t, u, v][h, u, v, w, t][h, t, w, u, v][h, t, u, v, w][h, u, v, t, w][h, w, u, v, t]\\&[h, t, u, v, w, t][h, t, w, t, u, v][h, u, v, t, t, w]^{-1}[h, w, u, v, t, t]^{-1}.\end{aligned}$$

First suppose that $u = a$ and $v = b$. Further, suppose that $w = [c, x]$. For this case we first suppose that $t = x$. Then

$$\begin{aligned}S =&[h, [c, x], x, a, b][h, a, b, [c, x], x][h, x, [c, x], a, b][h, x, a, b, [c, x]]\\&[h, a, b, x, [c, x]][h, [c, x], a, b, x].\end{aligned}$$

Commuting by y or y, y gives, in terms of (A.1),

$$\begin{aligned}&[h, [c, x], x, a, b, y, y][h, a, b, [c, x], x, y, y][h, x, [c, x], a, b, y, y][h, x, a, b, [c, x], y, y]\\&[h, a, b, x, [c, x], y, y][h, [c, x], a, b, x, y, y] = zz^{-4}z^4zz^{-1}z^{-1} = 1.\end{aligned}$$

Next suppose that $t = [x, y]$. Then

$$\begin{aligned}S =&[h, [c, x], [x, y], a, b][h, a, b, [c, x], [x, y]][h, [x, y], [c, x], a, b][h, [x, y], a, b, [c, x]]\\&[h, a, b, [x, y], [c, x]][h, [c, x], a, b, [x, y]].\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h, a, b, [c, x], y, x, y][h, y, x, y, [c, x], b, a]^{-1})^{-9} = (z^3z^{-3})^{-9} = 1.$$

Instead commuting by y gives

$$\begin{aligned}&[h, [c, x], [x, y], a, b, y][h, a, b, [c, x], [x, y], y][h, [x, y], [c, x], a, b, y][h, [x, y], a, b, [c, x], y]\\&[h, a, b, [x, y], [c, x], y][h, [c, x], a, b, [x, y], y] = z^5z^{-7}z^{-7}z^2z^5z^2 = 1.\end{aligned}$$

Now suppose that $t = xy$. Then

$$\begin{aligned}
S =&[h,[c,x],x,y,a,b][h,a,b,[c,x],x,y][h,x,y,[c,x],a,b][h,x,y,a,b,[c,x]]\\
&[h,a,b,x,y,[c,x]][h,[c,x],a,b,x,y][h,x,a,b,[c,x],y][h,y,a,b,[c,x],x]\\
&[h,y,a,b,[c,x],x,y][h,x,y,a,b,[c,x],y][h,x,[c,x],y,a,b][h,y,[c,x],x,a,b]\\
&[h,y,[c,x],x,y,a,b][h,x,y,[c,x],y,a,b][h,a,b,x,y,[c,x]]^{-1}[h,a,b,y,x,[c,x]]^{-1}\\
&[h,a,b,y,x,y,[c,x]]^{-1}[h,a,b,x,y,y,[c,x]]^{-1}[h,[c,x],a,b,x,y]^{-1}[h,[c,x],a,b,y,x]^{-1}\\
&[h,[c,x],a,b,y,x,y]^{-1}[h,[c,x],a,b,x,y,y]^{-1}\\
=&[h,[c,x],x,y,a,b][h,a,b,[c,x],x,y][h,x,y,[c,x],a,b][h,x,y,a,b,[c,x]]\\
&[h,x,a,b,[c,x],y][h,y,a,b,[c,x],x][h,x,[c,x],y,a,b][h,y,[c,x],x,a,b]\\
&[h,a,b,y,x,[c,x]]^{-1}[h,[c,x],a,b,y,x]^{-1}z^{12}.
\end{aligned}$$

In terms of (A.1) this gives

$$\begin{aligned}
&[h,[c,x],x,y,y,a,b][h,a,b,[c,x],x,y,y][h,x,y,y,[c,x],a,b][h,x,y,y,a,b,[c,x]]\\
&[h,x,a,b,[c,x],y,y][h,y,y,a,b,[c,x],x][h,x,[c,x],y,y,a,b][h,y,y,[c,x],x,a,b]\\
&[h,a,b,y,y,x,[c,x]]^{-1}[h,[c,x],a,b,y,y,x]^{-1}z^{12}\\
=&z^{-4}z^{-4}z^{-1}z^{-4}zz^{-4}z^{-1}zz^{-4}z^{-4}z^{24} = 1.
\end{aligned}$$

Instead commuting by y gives

$$\begin{aligned}
&[h,[c,x],x,y,a,b,y][h,a,b,[c,x],x,y,y][h,x,y,[c,x],a,b,y][h,x,y,a,b,[c,x],y]\\
&[h,x,a,b,[c,x],y,y][h,y,a,b,[c,x],x,y][h,x,[c,x],y,a,b,y][h,y,[c,x],x,a,b,y]\\
&[h,a,b,y,x,[c,x],y]^{-1}[h,[c,x],a,b,y,x,y]^{-1} = z^3z^{-4}z^{-3}z^3zz^{-4}z^{-3}zz^3z^3 = 1.
\end{aligned}$$

Now suppose instead that $w = [c, y]$. First consider $t = x$. Then

$$\begin{aligned}
S =&[h,[c,y],x,a,b][h,a,b,[c,y],x][h,x,[c,y],a,b][h,x,a,b,[c,y]][h,a,b,x,[c,y]]\\
&[h,[c,y],a,b,x][h,x,a,b,[c,y],x][h,x,[c,y],x,a,b][h,a,b,x,x,[c,y]]^{-1}\\
&[h,[c,y],a,b,x,x]^{-1}.
\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$\begin{aligned}
&([h,a,b,y,c,y,x,x]^3[h,y,c,y,x,x,b,a]^{-3}[h,x,a,b,y,c,y,x]^2[h,x,y,c,y,x,a,b]^2\\
&[h,a,b,x,x,y,c,y]^{-2}[h,y,c,y,a,b,x,x]^{-2})^{-3} = (z^3z^3z^2z^{-4}z^{-2}z^{-2})^{-3} = 1.
\end{aligned}$$

Instead commuting by x first gives, in terms of (A.1),

$$([h,y,c,y,x,a,b,x][h,a,b,y,c,y,x,x][h,x,y,c,y,a,b,x][h,x,a,b,y,c,y,x]$$
$$[h,a,b,x,y,c,y,x][h,y,c,y,a,b,x,x])^{-3} = (z^{-2}zzzz^{-2}z)^{-3} = 1.$$

Instead commuting by y gives

$$[h,[c,y],x,x,a,b,y][h,a,b,[c,y],x,x,y][h,x,x,[c,y],a,b,y][h,x,x,a,b,[c,y],y]$$
$$[h,a,b,x,x,[c,y],y][h,[c,y],a,b,x,x,y][h,x,a,b,[c,y],x,y]^2[h,x,[c,y],x,a,b,y]^2$$
$$[h,a,b,x,x,[c,y],y]^{-2}[h,[c,y],a,b,x,x,y]^{-2} = z^{-1}zz^{-1}z^{-4}zz^4z^6z^4z^{-2}z^{-8} = 1.$$

Commuting by x, y or by y, x gives

$$[h,[c,y],x,a,b,x,y][h,a,b,[c,y],x,x,y][h,x,[c,y],a,b,x,y][h,x,a,b,[c,y],x,y]$$
$$[h,a,b,x,[c,y],x,y][h,[c,y],a,b,x,x,y] = z^{-3}zz^{-3}z^3z^{-2}z^4 = 1,$$

$$[h,[c,y],x,a,b,y,x][h,a,b,[c,y],x,y,x][h,x,[c,y],a,b,y,x][h,x,a,b,[c,y],y,x]$$
$$[h,a,b,x,[c,y],y,x][h,[c,y],a,b,x,y,x] = z^2z^3z^{-1}z^{-4}z^3z^{-3} = 1.$$

Next consider $t = [x,y]$. This case follows from the case $w = [c,x]$ and $t = [x,y] = [y,x]^{-1}$ by swapping x and y. Now consider $t = xy$. Then

$$\begin{aligned}
S =&[h,[c,y],x,y,a,b][h,a,b,[c,y],x,y][h,x,y,[c,y],a,b][h,x,y,a,b,[c,y]]\\
&[h,a,b,x,y,[c,y]][h,[c,y],a,b,x,y][h,x,a,b,[c,y],y][h,y,a,b,[c,y],x]\\
&[h,x,a,b,[c,y],x,y][h,x,y,a,b,[c,y],x][h,x,[c,y],y,a,b][h,y,[c,y],x,a,b]\\
&[h,x,[c,y],x,y,a,b][h,x,y,[c,y],x,a,b][h,a,b,x,y,[c,y]]^{-1}[h,a,b,y,x,[c,y]]^{-1}\\
&[h,a,b,x,x,y,[c,y]]^{-1}[h,a,b,x,y,x,[c,y]]^{-1}[h,[c,y],a,b,x,y]^{-1}[h,[c,y],a,b,y,x]^{-1}\\
&[h,[c,y],a,b,x,x,y]^{-1}[h,[c,y],a,b,x,y,x]^{-1}\\
=&[h,[c,y],x,y,a,b][h,a,b,[c,y],x,y][h,x,y,[c,y],a,b][h,x,y,a,b,[c,y]]\\
&[h,x,a,b,[c,y],y][h,y,a,b,[c,y],x][h,x,[c,y],y,a,b][h,y,[c,y],x,a,b]\\
&[h,a,b,y,x,[c,y]]^{-1}[h,[c,y],a,b,y,x]^{-1}z^{-3}.
\end{aligned}$$

In terms of (A.1) this gives

$$
\begin{aligned}
&[h,[c,y],x,x,y,a,b][h,a,b,[c,y],x,x,y][h,x,x,y,[c,y],a,b][h,x,x,y,a,b,[c,y]]\\
&[h,x,x,a,b,[c,y],y][h,y,a,b,[c,y],x,x][h,x,x,[c,y],y,a,b][h,y,[c,y],x,x,a,b]\\
&[h,a,b,y,x,x,[c,y]]^{-1}[h,[c,y],a,b,y,x,x]^{-1}z^{-6}\\
=&zzz^{4}zz^{-4}zzz^{-1}zzz^{-6}=1.
\end{aligned}
$$

Instead commuting by x gives

$$
\begin{aligned}
&[h,[c,y],x,y,a,b,x][h,a,b,[c,y],x,y,x][h,x,y,[c,y],a,b,x][h,x,y,a,b,[c,y],x]\\
&[h,x,a,b,[c,y],y,x][h,y,a,b,[c,y],x,x][h,x,[c,y],y,a,b,x][h,y,[c,y],x,a,b,x]\\
&[h,a,b,y,x,[c,y],x]^{-1}[h,[c,y],a,b,y,x,x]^{-1}=z^{-2}z^{3}z^{4}zz^{-4}zzz^{-3}z^{-2}z=1.
\end{aligned}
$$

Finally for this u, v and w consider $t=x[x,y]$. Then

$$
\begin{aligned}
S=&[h,[c,y],x,[x,y],a,b][h,a,b,[c,y],x,[x,y]][h,x,[x,y],[c,y],a,b][h,x,[x,y],a,b,[c,y]]\\
&[h,a,b,x,[x,y],[c,y]][h,[c,y],a,b,x,[x,y]][h,x,a,b,[c,y],[x,y]][h,[x,y],a,b,[c,y],x]\\
&[h,x,[c,y],[x,y],a,b][h,[x,y],[c,y],x,a,b][h,a,b,x,[x,y],[c,y]]^{-1}\\
&[h,a,b,[x,y],x,[c,y]]^{-1}[h,[c,y],a,b,x,[x,y]]^{-1}[h,[c,y],a,b,[x,y],x]^{-1}\\
=&z^{-2}z^{-2}z^{7}z^{-2}z^{7}z^{7}z^{7}z^{-2}z^{-5}z^{-5}z^{-7}z^{2}z^{-7}z^{2}=1.
\end{aligned}
$$

Now consider $u=a$, $v=b$ and $w=[c,x,y]$. Suppose that $t=x$, then

$$
\begin{aligned}
S=&[h,[c,x,y],x,a,b][h,a,b,[c,x,y],x][h,x,[c,x,y],a,b][h,x,a,b,[c,x,y]]\\
&[h,a,b,x,[c,x,y]][h,[c,x,y],a,b,x].
\end{aligned}
$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$
\begin{aligned}
&([h,a,b,y,x,c,y,x][h,a,b,y,c,x,y,x]^{-1}[h,x,y,x,c,y,b,a]^{-1}[h,x,y,c,x,y,b,a])^{9}\\
=&(z^{0}z^{2}z^{-2}z^{0})^{9}=1.
\end{aligned}
$$

Instead commuting by y gives

$$
\begin{aligned}
&[h,[c,x,y],x,a,b,y][h,a,b,[c,x,y],x,y][h,x,[c,x,y],a,b,y][h,x,a,b,[c,x,y],y]\\
&[h,a,b,x,[c,x,y],y][h,[c,x,y],a,b,x,y]=z^{-3}z^{0}z^{0}z^{3}z^{-3}z^{3}=1.
\end{aligned}
$$

For $t = xy$,

$$\begin{aligned}S =&[h,[c,x,y],x,y,a,b][h,a,b,[c,x,y],x,y][h,x,y,[c,x,y],a,b][h,x,y,a,b,[c,x,y]]\\&[h,a,b,x,y,[c,x,y]][h,[c,x,y],a,b,x,y][h,x,a,b,[c,x,y],y][h,y,a,b,[c,x,y],x]\\&[h,x,[c,x,y],y,a,b][h,y,[c,x,y],x,a,b][h,a,b,x,y,[c,x,y]]^{-1}\\&[h,a,b,y,x,[c,x,y]]^{-1}[h,[c,x,y],a,b,x,y]^{-1}[h,[c,x,y],a,b,y,x]^{-1}\\=&z^0z^0z^3z^0z^3z^3z^3z^0z^{-3}z^{-3}z^{-3}z^0z^{-3}z^0 = 1.\end{aligned}$$

Now instead consider $w = [c, y, x]$. For $t = x$,

$$\begin{aligned}S =&[h,[c,y,x],x,a,b][h,a,b,[c,y,x],x][h,x,[c,y,x],a,b][h,x,a,b,[c,y,x]]\\&[h,a,b,x,[c,y,x]][h,[c,y,x],a,b,x].\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$\begin{aligned}&([h,a,b,x,y,c,y,x][h,a,b,y,c,y,x,x]^{-1}[h,x,x,y,c,y,b,a]^{-1}[h,x,y,c,y,x,b,a])^9\\=&(z^{-2}z^{-1}zz^2)^9 = 1.\end{aligned}$$

Instead commuting by y gives

$$\begin{aligned}&[h,[c,y,x],x,a,b,y][h,a,b,[c,y,x],x,y][h,x,[c,y,x],a,b,y][h,x,a,b,[c,y,x],y]\\&[h,a,b,x,[c,y,x],y][h,[c,y,x],a,b,x,y] = z^{-3}z^3z^3z^0z^{-3}z^0 = 1.\end{aligned}$$

For $t = xy$,

$$\begin{aligned}S =&[h,[c,y,x],x,y,a,b][h,a,b,[c,y,x],x,y][h,x,y,[c,y,x],a,b][h,x,y,a,b,[c,y,x]]\\&[h,a,b,x,y,[c,y,x]][h,[c,y,x],a,b,x,y][h,x,a,b,[c,y,x],y][h,y,a,b,[c,y,x],x]\\&[h,x,[c,y,x],y,a,b][h,y,[c,y,x],x,a,b][h,a,b,x,y,[c,y,x]]^{-1}\\&[h,a,b,y,x,[c,y,x]]^{-1}[h,[c,y,x],a,b,x,y]^{-1}[h,[c,y,x],a,b,y,x]^{-1}\\=&z^3z^3z^0z^3z^0z^0z^0z^3z^{-3}z^{-3}z^0z^{-3}z^0z^{-3} = 1.\end{aligned}$$

Now suppose that $u = [a, y]$ and $v = [b, y]$. For $w = c$ and $t = x$,

$$\begin{aligned}S =&[h,c,x,[a,y],[b,y]][h,[a,y],[b,y],c,x][h,x,c,[a,y],[b,y]][h,x,[a,y],[b,y],c]\\&[h,[a,y],[b,y],x,c][h,c,[a,y],[b,y],x][h,x,[a,y],[b,y],c,x][h,x,c,x,[a,y],[b,y]]\\&[h,[a,y],[b,y],x,x,c]^{-1}[h,c,[a,y],[b,y],x,x]^{-1}.\end{aligned}$$

Using (A.2) this gives, in terms of (A.1),

$$[h,[a,y],[b,y],c,x,x]^3[h,x,x,c,[b,y],[a,y]]^{-3}z^{-12}z^{24}z^{12}z^{12}=z^{-18}z^{-18}z^{36}=1.$$

Instead commuting by x gives

$$\begin{aligned}&[h,c,x,[a,y],[b,y],x][h,[a,y],[b,y],c,x,x][h,x,c,[a,y],[b,y],x][h,x,[a,y],[b,y],c,x]\\&[h,[a,y],[b,y],x,c,x][h,c,[a,y],[b,y],x,x]=z^{12}z^{-6}z^{-6}z^{-6}z^{12}z^{-6}=1.\end{aligned}$$

Now consider the case u and w are of the same form. Then

$$\begin{aligned}S=&[h,v,w,t,u]^2[h,w,t,u,v][h,u,v,w,t]^2[h,w,u,t,v]^{-1}[h,t,u,v,w][h,t,v,w,u]\\&[h,t,v,w,t,u][h,t,u,v,w,t][h,t,w,t,u,v][h,t,w,u,t,v]^{-1}\\&[h,v,w,t,t,u]^{-1}[h,u,v,w,t,t]^{-1}.\end{aligned}$$

Note that (A.2) becomes $[h,u,v,w,t]^6$. First suppose that $u=a$ and $w=c$. Further, suppose that $v=[b,x]$. For $t=x$,

$$\begin{aligned}S=&[h,[b,x],c,x,a]^2[h,c,x,a,[b,x]][h,a,[b,x],c,x]^2[h,c,a,x,[b,x]]^{-1}\\&[h,x,a,[b,x],c][h,x,[b,x],c,a].\end{aligned}$$

Commuting by y or y,y this gives, in terms of (A.1),

$$\begin{aligned}&[h,[b,x],c,x,a,y,y]^2[h,c,x,a,[b,x],y,y][h,a,[b,x],c,x,y,y]^2[h,c,a,x,[b,x],y,y]^{-1}\\&[h,x,a,[b,x],c,y,y][h,x,[b,x],c,a,y,y]=z^{-10}z^5z^0zz^0z^4=1.\end{aligned}$$

Now consider $t=[x,y]$. Then

$$\begin{aligned}S=&[h,[b,x],c,[x,y],a]^2[h,c,[x,y],a,[b,x]][h,a,[b,x],c,[x,y]]^2[h,c,a,[x,y],[b,x]]^{-1}\\&[h,[x,y],a,[b,x],c][h,[x,y],[b,x],c,a].\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1), $[h,a,[b,x],c,y,x,y]^{-18}=1$. Instead commuting by y gives

$$\begin{aligned}&[h,[b,x],c,[x,y],a,y]^2[h,c,[x,y],a,[b,x],y][h,a,[b,x],c,[x,y],y]^2\\&[h,c,a,[x,y],[b,x],y]^{-1}[h,[x,y],a,[b,x],c,y][h,[x,y],[b,x],c,a,y]\\&=z^8z^2z^0z^{-3}z^0z^{-7}=1.\end{aligned}$$

Next consider $t = xy$. Then

$$
\begin{aligned}
S =& [h,[b,x],c,x,y,a]^2[h,c,x,y,a,[b,x]][h,a,[b,x],c,x,y]^2[h,c,a,x,y,[b,x]]^{-1}\\
& [h,x,y,a,[b,x],c][h,x,y,[b,x],c,a][h,x,[b,x],c,y,a][h,y,[b,x],c,x,a]\\
& [h,y,[b,x],c,x,y,a][h,x,y,[b,x],c,y,a][h,x,a,[b,x],c,y][h,y,a,[b,x],c,x]\\
& [h,y,a,[b,x],c,x,y][h,x,y,a,[b,x],c,y][h,x,c,y,a,[b,x]][h,y,c,x,a,[b,x]]\\
& [h,y,c,x,y,a,[b,x]][h,x,y,c,y,a,[b,x]][h,x,c,a,y,[b,x]]^{-1}[h,y,c,a,x,[b,x]]^{-1}\\
& [h,y,c,a,x,y,[b,x]]^{-1}[h,x,y,c,a,y,[b,x]]^{-1}[h,[b,x],c,x,y,a]^{-1}[h,[b,x],c,y,x,a]^{-1}\\
& [h,[b,x],c,y,x,y,a]^{-1}[h,[b,x],c,x,y,y,a]^{-1}[h,a,[b,x],c,x,y]^{-1}[h,a,[b,x],c,y,x]^{-1}\\
& [h,a,[b,x],c,y,x,y]^{-1}[h,a,[b,x],c,x,y,y]^{-1}\\
=& [h,[b,x],c,x,y,a][h,c,x,y,a,[b,x]][h,a,[b,x],c,x,y][h,c,a,x,y,[b,x]]^{-1}\\
& [h,x,y,a,[b,x],c][h,x,y,[b,x],c,a][h,x,[b,x],c,y,a][h,y,[b,x],c,x,a]\\
& [h,x,a,[b,x],c,y][h,y,a,[b,x],c,x][h,x,c,y,a,[b,x]][h,y,c,x,a,[b,x]]\\
& [h,x,c,a,y,[b,x]]^{-1}[h,y,c,a,x,[b,x]]^{-1}[h,[b,x],c,y,x,a]^{-1}[h,a,[b,x],c,y,x]^{-1}.
\end{aligned}
$$

In terms of (A.1) this gives

$$
\begin{aligned}
& [h,[b,x],c,x,y,y,a][h,c,x,y,y,a,[b,x]][h,a,[b,x],c,x,y,y][h,c,a,x,y,y,[b,x]]^{-1}\\
& [h,x,y,y,a,[b,x],c][h,x,y,y,[b,x],c,a][h,x,[b,x],c,y,y,a][h,y,y,[b,x],c,x,a]\\
& [h,x,a,[b,x],c,y,y][h,y,y,a,[b,x],c,x][h,x,c,y,y,a,[b,x]][h,y,y,c,x,a,[b,x]]\\
& [h,x,c,a,y,y,[b,x]]^{-1}[h,y,y,c,a,x,[b,x]]^{-1}[h,[b,x],c,y,y,x,a]^{-1}\\
& [h,a,[b,x],c,y,y,x]^{-1} = z^0z^0z^0zz^0z^{-1}z^{-6}z^{-5}z^0z^0z^6z^5z^{-1}zz^0z^0 = 1.
\end{aligned}
$$

Instead commuting by y gives

$$
\begin{aligned}
& [h,[b,x],c,x,y,a,y][h,c,x,y,a,[b,x],y][h,a,[b,x],c,x,y,y][h,c,a,x,y,[b,x],y]^{-1}\\
& [h,x,y,a,[b,x],c,y][h,x,y,[b,x],c,a,y][h,x,[b,x],c,y,a,y][h,y,[b,x],c,x,a,y]\\
& [h,x,a,[b,x],c,y,y][h,y,a,[b,x],c,x,y][h,x,c,y,a,[b,x],y][h,y,c,x,a,[b,x],y]\\
& [h,x,c,a,y,[b,x],y]^{-1}[h,y,c,a,x,[b,x],y]^{-1}[h,[b,x],c,y,x,a,y]^{-1}\\
& [h,a,[b,x],c,y,x,y]^{-1} = z^5z^{-1}z^0z^{-2}z^0z^{-3}z^2z^{-5}z^0z^0z^{-3}z^5z^2zz^{-1}z^0 = 1.
\end{aligned}
$$

Next suppose that $v = [b, y]$. For $t = x$,

$$\begin{aligned}S =&[h,[b,y],c,x,a]^2[h,c,x,a,[b,y]][h,a,[b,y],c,x]^2[h,c,a,x,[b,y]]^{-1}[h,x,a,[b,y],c]\\&[h,x,[b,y],c,a][h,x,[b,y],c,x,a][h,x,a,[b,y],c,x][h,x,c,x,a,[b,y]]\\&[h,x,c,a,x,[b,y]]^{-1}[h,[b,y],c,x,x,a]^{-1}[h,a,[b,y],c,x,x]^{-1}.\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$\begin{aligned}&([h,a,y,b,y,c,x,x]^6[h,x,y,b,y,c,x,a]^2[h,x,a,y,b,y,c,x]^2[h,x,c,x,a,y,b,y]^2\\&[h,x,c,a,x,y,b,y]^{-2}[h,y,b,y,c,x,x,a]^{-2}[h,a,y,b,y,c,x,x]^{-2})^{-3}\\=&(z^{-24}z^6z^{-8}z^6z^4z^8z^8)^{-3} = 1.\end{aligned}$$

Instead commuting by x first gives

$$\begin{aligned}&([h,y,b,y,c,x,a,x]^2[h,c,x,a,y,b,y,x][h,a,y,b,y,c,x,x]^2[h,c,a,x,y,b,y,x]^{-1}\\&[h,x,a,y,b,y,c,x][h,x,y,b,y,c,a,x] = z^6z^3z^{-8}z^2z^{-4}z = 1.\end{aligned}$$

Instead commuting by y first gives

$$\begin{aligned}&[h,[b,y],c,x,x,a,y]^2[h,c,x,x,a,[b,y],y][h,a,[b,y],c,x,x,y]^2[h,c,a,x,x,[b,y],y]^{-1}\\&[h,x,x,a,[b,y],c,y][h,x,x,[b,y],c,a,y][h,x,[b,y],c,x,a,y]^2[h,x,a,[b,y],c,x,y]^2\\&[h,x,c,x,a,[b,y],y]^2[h,x,c,a,x,[b,y],y]^{-2}[h,[b,y],c,x,x,a,y]^{-2}\\&[h,a,[b,y],c,x,x,y]^{-2} = z^{-12}z^6z^0z^{-1}z^0z^{-1}z^6z^0z^{-4}z^{-6}z^{12}z^0 = 1.\end{aligned}$$

Instead commuting by x, y or by y, x gives

$$\begin{aligned}&[h,[b,y],c,x,a,x,y]^2[h,c,x,a,[b,y],x,y][h,a,[b,y],c,x,x,y]^2[h,c,a,x,[b,y],x,y]^{-1}\\&[h,x,a,[b,y],c,x,y][h,x,[b,y],c,a,x,y] = z^4z^{-3}z^0z^2z^0z^{-3} = 1,\end{aligned}$$

$$\begin{aligned}&[h,[b,y],c,x,a,y,x]^2[h,c,x,a,[b,y],y,x][h,a,[b,y],c,x,y,x]^2[h,c,a,x,[b,y],y,x]^{-1}\\&[h,x,a,[b,y],c,y,x][h,x,[b,y],c,a,y,x] = z^6z^{-2}z^0z^{-3}z^0z^{-1} = 1.\end{aligned}$$

Next consider $t = [x, y]$. This follows from the case $v = [b, x]$ by swapping x and y.

Now consider $t = xy$. Then

$$
\begin{aligned}
S =& [h,[b,y],c,x,y,a]^2[h,c,x,y,a,[b,y]][h,a,[b,y],c,x,y]^2[h,c,a,x,y,[b,y]]^{-1} \\
& [h,x,y,a,[b,y],c][h,x,y,[b,y],c,a][h,x,[b,y],c,y,a][h,y,[b,y],c,x,a] \\
& [h,x,[b,y],c,x,y,a][h,x,y,[b,y],c,x,a][h,x,a,[b,y],c,y][h,y,a,[b,y],c,x] \\
& [h,x,a,[b,y],c,x,y][h,x,y,a,[b,y],c,x][h,x,c,y,a,[b,y]][h,y,c,x,a,[b,y]] \\
& [h,x,c,x,y,a,[b,y]][h,x,y,c,x,a,[b,y]][h,x,c,a,y,[b,y]]^{-1}[h,y,c,a,x,[b,y]]^{-1} \\
& [h,x,c,a,x,y,[b,y]]^{-1}[h,x,y,c,a,x,[b,y]]^{-1}[h,[b,y],c,x,y,a]^{-1}[h,[b,y],c,y,x,a]^{-1} \\
& [h,[b,y],c,x,x,y,a]^{-1}[h,[b,y],c,x,y,x,a]^{-1}[h,a,[b,y],c,x,y]^{-1}[h,a,[b,y],c,y,x]^{-1} \\
& [h,a,[b,y],c,x,x,y]^{-1}[h,a,[b,y],c,x,y,x]^{-1} \\
=& [h,[b,y],c,x,y,a][h,c,x,y,a,[b,y]][h,a,[b,y],c,x,y][h,c,a,x,y,[b,y]]^{-1} \\
& [h,x,y,a,[b,y],c][h,x,y,[b,y],c,a][h,x,[b,y],c,y,a][h,y,[b,y],c,x,a] \\
& [h,x,a,[b,y],c,y][h,y,a,[b,y],c,x][h,x,c,y,a,[b,y]][h,y,c,x,a,[b,y]] \\
& [h,x,c,a,y,[b,y]]^{-1}[h,y,c,a,x,[b,y]]^{-1}[h,[b,y],c,y,x,a]^{-1}[h,a,[b,y],c,y,x]^{-1}.
\end{aligned}
$$

In terms of (A.1) this gives

$$
\begin{aligned}
& [h,[b,y],c,x,x,y,a][h,c,x,x,y,a,[b,y]][h,a,[b,y],c,x,x,y][h,c,a,x,x,y,[b,y]]^{-1} \\
& [h,x,x,y,a,[b,y],c][h,x,x,y,[b,y],c,a][h,x,x,[b,y],c,y,a][h,y,[b,y],c,x,x,a] \\
& [h,x,x,a,[b,y],c,y][h,y,a,[b,y],c,x,x][h,x,x,c,y,a,[b,y]][h,y,c,x,x,a,[b,y]] \\
& [h,x,x,c,a,y,[b,y]]^{-1}[h,y,c,a,x,x,[b,y]]^{-1}[h,[b,y],c,y,x,x,a]^{-1} \\
& [h,a,[b,y],c,y,x,x]^{-1} = z^0z^0z^0zz^0z^{-1}z^{-5}z^{-6}z^0z^0z^5z^6zz^{-1}z^0z^0 = 1.
\end{aligned}
$$

Instead commuting by x gives

$$
\begin{aligned}
& [h,[b,y],c,x,y,a,x][h,c,x,y,a,[b,y],x][h,a,[b,y],c,x,y,x][h,c,a,x,y,[b,y],x]^{-1} \\
& [h,x,y,a,[b,y],c,x][h,x,y,[b,y],c,a,x][h,x,[b,y],c,y,a,x][h,y,[b,y],c,x,a,x] \\
& [h,x,a,[b,y],c,y,x][h,y,a,[b,y],c,x,x][h,x,c,y,a,[b,y],x][h,y,c,x,a,[b,y],x] \\
& [h,x,c,a,y,[b,y],x]^{-1}[h,y,c,a,x,[b,y],x]^{-1}[h,[b,y],c,y,x,a,x]^{-1} \\
& [h,a,[b,y],c,y,x,x]^{-1} = zz^{-5}z^0z^3z^0z^4z^{-5}z^2z^0z^0z^5z^{-3}zz^2z^{-5}z^0 = 1.
\end{aligned}
$$

For $t = x[x, y]$,

$$\begin{aligned}
S =&[h, [b, y], c, x, [x, y], a][h, c, x, [x, y], a, [b, y]][h, a, [b, y], c, x, [x, y]] \\
&[h, c, a, x, [x, y], [b, y]]^{-1}[h, x, [x, y], a, [b, y], c][h, x, [x, y], [b, y], c, a] \\
&[h, x, [b, y], c, [x, y], a][h, [x, y], [b, y], c, x, a][h, x, a, [b, y], c, [x, y]][h, [x, y], a, [b, y], c, x] \\
&[h, x, c, [x, y], a, [b, y]][[h, [x, y], c, x, a, [b, y]]h, x, c, a, [x, y], [b, y]]^{-1} \\
&[h, [x, y], c, a, x, [b, y]]^{-1}[h, [b, y], c, [x, y], x, a]^{-1}[h, a, [b, y], c, [x, y], x]^{-1} \\
=&z^0 z^0 z^0 z^{-7} z^0 z^7 z^{-4} z^{-1} z^0 z^0 z^{-4} z^{-1} z^5 z^5 z^0 z^0 = 1.
\end{aligned}$$

Next consider $v = [b, x, y]$. Suppose that $t = x$. Then

$$\begin{aligned}
S =&[h, [b, x, y], c, x, a]^2[h, c, x, a, [b, x, y]][h, a, [b, x, y], c, x]^2[h, c, a, x, [b, x, y]]^{-1} \\
&[h, x, a, [b, x, y], c][h, x, [b, x, y], c, a].
\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h, a, y, x, b, y, c, x]^6[h, a, y, b, x, y, c, x]^{-6})^3 = (z^6 z^{-6})^3 = 1.$$

Instead commuting by y gives

$$\begin{aligned}
&[h, [b, x, y], c, x, a, y]^2[h, c, x, a, [b, x, y], y][h, a, [b, x, y], c, x, y]^2[h, c, a, x, [b, x, y], y]^{-1} \\
&[h, x, a, [b, x, y], c, y][h, x, [b, x, y], c, a, y] = z^{10} z^5 z^{-12} z^3 z^{-6} z^0 = 1.
\end{aligned}$$

Now suppose that $t = xy$. Then

$$\begin{aligned}
S =&[h, [b, x, y], c, x, y, a]^2[h, c, x, y, a, [b, x, y]][h, a, [b, x, y], c, x, y]^2[h, c, a, x, y, [b, x, y]]^{-1} \\
&[h, x, y, a, [b, x, y], c][h, x, y, [b, x, y], c, a][h, x, [b, x, y], c, y, a][h, y, [b, x, y], c, x, a] \\
&[h, x, a, [b, x, y], c, y][h, y, a, [b, x, y], c, x][h, x, c, y, a, [b, x, y]][h, y, c, x, a, [b, x, y]] \\
&[h, x, c, a, y, [b, x, y]]^{-1}[h, y, c, a, x, [b, x, y]]^{-1}[h, [b, x, y], c, x, y, a]^{-1} \\
&[h, [b, x, y], c, y, x, a]^{-1}[h, a, [b, x, y], c, x, y]^{-1}[h, a, [b, x, y], c, y, x]^{-1} \\
=&z^{-12} z^{-6} z^{-12} z^{-3} z^{-6} z^3 z^4 z^5 z^{-6} z^{-6} z^4 z^5 z^3 z^3 z^6 z^6 z^6 z^6 = 1.
\end{aligned}$$

Now consider $w = [b, y, x]$. For $t = x$,

$$\begin{aligned}
S =&[h, [b, y, x], c, x, a]^2[h, c, x, a, [b, y, x]][h, a, [b, y, x], c, x]^2[h, c, a, x, [b, y, x]]^{-1} \\
&[h, x, a, [b, y, x], c][h, x, [b, y, x], c, a].
\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h, a, x, y, b, y, c, x]^3 [h, a, y, b, y, x, c, x]^{-3})^3 = (z^9 z^{-9})^3 = 1.$$

Instead commuting by y gives

$$\begin{aligned}&[h, [b, y, x], c, x, a, y]^2 [h, c, x, a, [b, y, x], y][h, a, [b, y, x], c, x, y]^2 [h, c, a, x, [b, y, x], y]^{-1} \\ &[h, x, a, [b, y, x], c, y][h, x, [b, y, x], c, a, y] = z^8 z^4 z^{-12} z^3 z^{-6} z^3 = 1.\end{aligned}$$

For $t = xy$,

$$\begin{aligned}S =&[h, [b, y, x], c, x, y, a]^2 [h, c, x, y, a, [b, y, x]][h, a, [b, y, x], c, x, y]^2 [h, c, a, x, y, [b, y, x]]^{-1} \\ &[h, x, y, a, [b, y, x], c][h, x, y, [b, y, x], c, a][h, x, [b, y, x], c, y, a][h, y, [b, y, x], c, x, a] \\ &[h, x, a, [b, y, x], c, y][h, y, a, [b, y, x], c, x][h, x, c, y, a, [b, y, x]][h, y, c, x, a, [b, y, x]] \\ &[h, x, c, a, y, [b, y, x]]^{-1} [h, y, c, a, x, [b, y, x]]^{-1} [h, [b, y, x], c, x, y, a]^{-1} \\ &[h, [b, y, x], c, y, x, a]^{-1} [h, a, [b, y, x], c, x, y]^{-1} [h, a, [b, y, x], c, y, x]^{-1} \\ =&z^{-12} z^{-6} z^{-12} z^0 z^{-6} z^0 z^5 z^4 z^{-6} z^{-6} z^5 z^4 z^3 z^3 z^6 z^6 z^6 z^6 = 1.\end{aligned}$$

Now suppose that $u = [a, y]$, $v = b$ and $w = [c, y]$. The only case to consider here is $t = x$. Then

$$\begin{aligned}S =&[h, b, [c, y], x, [a, y]]^2 [h, [c, y], x, [a, y], b][h, [a, y], b, [c, y], x]^2 [h, [c, y], [a, y], x, b]^{-1} \\ &[h, x, [a, y], b, [c, y]][h, x, b, [c, y], [a, y]][h, x, b, [c, y], x, [a, y]][h, x, [a, y], b, [c, y], x] \\ &[h, x, [c, y], x, [a, y], b][h, x, [c, y], [a, y], x, b]^{-1} [h, b, [c, y], x, x, [a, y]]^{-1} \\ &[h, [a, y], b, [c, y], x, x]^{-1}.\end{aligned}$$

In terms of (A.1) this gives, using (A.2),

$$[h, [a, y], b, [c, y], x, x]^6 z^4 z^8 z^4 z^{-24} z^{-8} z^{-8} = z^{24} z^{-24} = 1.$$

Instead commuting by x gives

$$\begin{aligned}&[h, b, [c, y], x, [a, y], x]^2 [h, [c, y], x, [a, y], b, x][h, [a, y], b, [c, y], x, x]^2 \\ &[h, [c, y], [a, y], x, b, x]^{-1} [h, x, [a, y], b, [c, y], x][h, x, b, [c, y], [a, y], x] \\ =&z^4 z^2 z^8 z^{-12} z^4 z^{-6} = 1.\end{aligned}$$

This finishes the case u and w are of the same form.

Now suppose that v and w are of the same form. Then

$$\begin{aligned}S =&[h,v,w,t,u][h,w,t,u,v][h,u,v,w,t][h,u,v,t,w]^{-1}[h,t,v,w,u][h,t,w,u,v]\\&[h,w,u,t,v][h,v,w,u,t][h,t,v,w,t,u][h,t,u,v,w,t][h,t,w,t,u,v]\\&[h,t,u,v,t,w]^{-1}[h,w,u,t,t,v]^{-1}[h,v,w,u,t,t]^{-1}.\end{aligned}$$

First consider $v = b$ and $w = c$. Suppose that $u = [a, x]$ and $t = x$. Then

$$\begin{aligned}S =&[h,b,c,x,[a,x]][h,c,x,[a,x],b][h,[a,x],b,c,x][h,[a,x],b,x,c]^{-1}\\&[h,x,b,c,[a,x]][h,x,c,[a,x],b][h,c,[a,x],x,b][h,b,c,[a,x],x].\end{aligned}$$

Commuting by y or y, y gives, in terms of (A.1),

$$\begin{aligned}&[h,b,c,x,[a,x],y,y][h,c,x,[a,x],b,y,y][h,[a,x],b,c,x,y,y][h,[a,x],b,x,c,y,y]^{-1}\\&[h,x,b,c,[a,x],y,y][h,x,c,[a,x],b,y,y][h,c,[a,x],x,b,y,y][h,b,c,[a,x],x,y,y]\\=&z^{-1}z^{-6}z^{-1}z^{5}zz^{0}z^{6}z^{-4} = 1.\end{aligned}$$

Next suppose that $t = [x, y]$. Then

$$\begin{aligned}S =&[h,b,c,[x,y],[a,x]][h,c,[x,y],[a,x],b][h,[a,x],b,c,[x,y]][h,[a,x],b,[x,y],c]^{-1}\\&[h,[x,y],b,c,[a,x]][h,[x,y],c,[a,x],b][h,c,[a,x],[x,y],b][h,b,c,[a,x],[x,y]].\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h,[a,x],b,c,y,x,y]^{3}[h,y,x,y,c,b,[a,x]]^{-3})^{-3} = (z^{-9}z^{9})^{-3} = 1.$$

Instead commuting by y gives

$$\begin{aligned}&[h,b,c,[x,y],[a,x],y][h,c,[x,y],[a,x],b,y][h,[a,x],b,c,[x,y],y][h,[a,x],b,[x,y],c,y]^{-1}\\&[h,[x,y],b,c,[a,x],y][h,[x,y],c,[a,x],b,y][h,c,[a,x],[x,y],b,y][h,b,c,[a,x],[x,y],y]\\=&z^{5}zz^{2}z^{-4}z^{2}z^{0}zz^{-7} = 1.\end{aligned}$$

Now suppose that $t = xy$. Then

$$\begin{aligned}
S =& [h, b, c, x, y, [a, x]][h, c, x, y, [a, x], b][h, [a, x], b, c, x, y][h, [a, x], b, x, y, c]^{-1} \\
& [h, x, y, b, c, [a, x]][h, x, y, c, [a, x], b][h, c, [a, x], x, y, b][h, b, c, [a, x], x, y] \\
& [h, x, b, c, y, [a, x]][h, y, b, c, x, [a, x]][h, y, b, c, x, y, [a, x]][h, x, y, b, c, y, [a, x]] \\
& [h, x, [a, x], b, c, y][h, y, [a, x], b, c, x][h, y, [a, x], b, c, x, y][h, x, y, [a, x], b, c, y] \\
& [h, x, c, y, [a, x], b][h, y, c, x, [a, x], b][h, y, c, x, y, [a, x], b][h, x, y, c, y, [a, x], b] \\
& [h, x, [a, x], b, y, c]^{-1}[h, y, [a, x], b, x, c]^{-1}[h, y, [a, x], b, x, y, c]^{-1}[h, x, y, [a, x], b, y, c]^{-1} \\
& [h, c, [a, x], x, y, b]^{-1}[h, c, [a, x], y, x, b]^{-1}[h, c, [a, x], y, x, y, b]^{-1}[h, c, [a, x], x, y, y, b]^{-1} \\
& [h, b, c, [a, x], x, y]^{-1}[h, b, c, [a, x], y, x]^{-1}[h, b, c, [a, x], y, x, y]^{-1}[h, b, c, [a, x], x, y, y]^{-1} \\
=& [h, b, c, x, y, [a, x]][h, c, x, y, [a, x], b][h, [a, x], b, c, x, y][h, [a, x], b, x, y, c]^{-1} \\
& [h, x, y, b, c, [a, x]][h, x, y, c, [a, x], b][h, x, b, c, y, [a, x]][h, y, b, c, x, [a, x]] \\
& [h, x, [a, x], b, c, y][h, y, [a, x], b, c, x][h, x, c, y, [a, x], b][h, y, c, x, [a, x], b] \\
& [h, x, [a, x], b, y, c]^{-1}[h, y, [a, x], b, x, c]^{-1}[h, c, [a, x], y, x, b]^{-1}[h, b, c, [a, x], y, x]^{-1}z^3.
\end{aligned}$$

In terms of (A.1) this gives

$$\begin{aligned}
& [h, b, c, x, y, y, [a, x]][h, c, x, y, y, [a, x], b][h, [a, x], b, c, x, y, y][h, [a, x], b, x, y, y, c]^{-1} \\
& [h, x, y, y, b, c, [a, x]][h, x, y, y, c, [a, x], b][h, x, b, c, y, y, [a, x]][h, y, y, b, c, x, [a, x]] \\
& [h, x, [a, x], b, c, y, y][h, y, y, [a, x], b, c, x][h, x, c, y, y, [a, x], b][h, y, y, c, x, [a, x], b] \\
& [h, x, [a, x], b, y, y, c]^{-1}[h, y, y, [a, x], b, x, c]^{-1}[h, c, [a, x], y, y, x, b]^{-1} \\
& [h, b, c, [a, x], y, y, x]^{-1}z^6 = z^{-1}z^{-1}z^{-1}z^0z^{-4}z^0zz^{-1}z^4z^{-1}z^{-5}z^{-6}z^6z^5z^{-1}z^{-1}z^6 = 1.
\end{aligned}$$

Commuting S by y gives

$$\begin{aligned}
& [h, b, c, x, y, [a, x], y][h, c, x, y, [a, x], b, y][h, [a, x], b, c, x, y, y][h, [a, x], b, x, y, c, y]^{-1} \\
& [h, x, y, b, c, [a, x], y][h, x, y, c, [a, x], b, y][h, x, b, c, y, [a, x], y][h, y, b, c, x, [a, x], y] \\
& [h, x, [a, x], b, c, y, y][h, y, [a, x], b, c, x, y][h, x, c, y, [a, x], b, y][h, y, c, x, [a, x], b, y] \\
& [h, x, [a, x], b, y, c, y]^{-1}[h, y, [a, x], b, x, c, y]^{-1}[h, c, [a, x], y, x, b, y]^{-1} \\
& [h, b, c, [a, x], y, x, y]^{-1} = z^2z^3z^{-1}z^{-5}z^3z^0z^{-2}z^{-1}z^4z^{-1}zz^{-6}z^{-2}z^5z^3z^{-3} = 1.
\end{aligned}$$

Now suppose that $u = [a, y]$ and $t = x$. Then

$$\begin{aligned} S =&[h, b, c, x, [a, y]][h, c, x, [a, y], b][h, [a, y], b, c, x][h, [a, y], b, x, c]^{-1}[h, x, b, c, [a, y]] \\ &[h, x, c, [a, y], b][h, c, [a, y], x, b][h, b, c, [a, y], x][h, x, b, c, x, [a, y]][h, x, [a, y], b, c, x] \\ &[h, x, c, x, [a, y], b][h, x, [a, y], b, x, c]^{-1}[h, c, [a, y], x, x, b]^{-1}[h, b, c, [a, y], x, x]^{-1}. \end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$\begin{aligned} &([h, y, a, y, b, c, x, x]^3[h, x, x, c, b, y, a, y]^{-3}[h, x, b, c, x, y, a, y]^2[h, x, y, a, y, b, c, x]^2 \\ &\ [h, x, c, x, y, a, y, b]^2[h, x, y, a, y, b, x, c]^{-2}[h, c, y, a, y, x, x, b]^{-2}[h, b, c, y, a, y, x, x]^{-2})^{-3} \\ =&(z^3z^3z^{-4}z^2z^6z^{-6}z^{-2}z^{-2})^{-3} = 1. \end{aligned}$$

Instead commuting first by x gives

$$\begin{aligned} &([h, b, c, x, y, a, y, x][h, c, x, y, a, y, b, x][h, y, a, y, b, c, x, x][h, y, a, y, b, x, c, x]^{-1} \\ &\ [h, x, b, c, y, a, y, x][h, x, c, y, a, y, b, x][h, c, y, a, y, x, b, x][h, b, c, y, a, y, x, x])^{-3} \\ &= (z^{-2}z^3zz^{-3}zz^{-4}z^3z)^{-3} = 1. \end{aligned}$$

Instead commuting first by y gives

$$\begin{aligned} &[h, b, c, x, x, [a, y], y][h, c, x, x, [a, y], b, y][h, [a, y], b, c, x, x, y][h, [a, y], b, x, x, c, y]^{-1} \\ &[h, x, x, b, c, [a, y], y][h, x, x, c, [a, y], b, y][h, c, [a, y], x, x, b, y][h, b, c, [a, y], x, x, y] \\ &[h, x, b, c, x, [a, y], y]^2[h, x, [a, y], b, c, x, y]^2[h, x, c, x, [a, y], b, y]^2[h, x, [a, y], b, x, c, y]^{-2} \\ &[h, c, [a, y], x, x, b, y]^{-2}[h, b, c, [a, y], x, x, y]^{-2} \\ =&zz^{-5}z^4z^6z^{-4}z^0z^5zz^6z^{-6}z^{10}z^{-6}z^{-10}z^{-2} = 1. \end{aligned}$$

Instead commuting by x, y or y, x gives

$$\begin{aligned} &[h, b, c, x, [a, y], x, y][h, c, x, [a, y], b, x, y][h, [a, y], b, c, x, x, y][h, [a, y], b, x, c, x, y]^{-1} \\ &[h, x, b, c, [a, y], x, y][h, x, c, [a, y], b, x, y][h, c, [a, y], x, b, x, y][h, b, c, [a, y], x, x, y] \\ =&z^{-2}zz^4z^{-2}z^3z^0z^{-5}z = 1, \end{aligned}$$

$$\begin{aligned} &[h, b, c, x, [a, y], y, x][h, c, x, [a, y], b, y, x][h, [a, y], b, c, x, y, x][h, [a, y], b, x, c, y, x]^{-1} \\ &[h, x, b, c, [a, y], y, x][h, x, c, [a, y], b, y, x][h, c, [a, y], x, b, y, x][h, b, c, [a, y], x, y, x] \\ =&z^3z^5z^{-3}z^{-3}z^{-4}z^0z^{-1}z^3 = 1. \end{aligned}$$

Next suppose that $t = [x, y]$. This follows from the case $u = [a, x]$ by swapping x and

y. Now suppose that $t = xy$. Then

$$
\begin{aligned}
S =&[h,b,c,x,y,[a,y]][h,c,x,y,[a,y],b][h,[a,y],b,c,x,y][h,[a,y],b,x,y,c]^{-1}\\
&[h,x,y,b,c,[a,y]][h,x,y,c,[a,y],b][h,c,[a,y],x,y,b][h,b,c,[a,y],x,y]\\
&[h,x,b,c,y,[a,y]][h,y,b,c,x,[a,y]][h,x,b,c,x,y,[a,y]][h,x,y,b,c,x,[a,y]]\\
&[h,x,[a,y],b,c,y][h,y,[a,y],b,c,x][h,x,[a,y],b,c,x,y][h,x,y,[a,y],b,c,x]\\
&[h,x,c,y,[a,y],b][h,y,c,x,[a,y],b][h,x,c,x,y,[a,y],b][h,x,y,c,x,[a,y],b]\\
&[h,x,[a,y],b,y,c]^{-1}[h,y,[a,y],b,x,c]^{-1}[h,x,[a,y],b,x,y,c]^{-1}[h,x,y,[a,y],b,x,c]^{-1}\\
&[h,c,[a,y],x,y,b]^{-1}[h,c,[a,y],y,x,b]^{-1}[h,c,[a,y],x,x,y,b]^{-1}[h,c,[a,y],x,y,x,b]^{-1}\\
&[h,b,c,[a,y],x,y]^{-1}[h,b,c,[a,y],y,x]^{-1}[h,b,c,[a,y],x,x,y]^{-1}[h,b,c,[a,y],x,y,x]^{-1}\\
=&[h,b,c,x,y,[a,y]][h,c,x,y,[a,y],b][h,[a,y],b,c,x,y][h,[a,y],b,x,y,c]^{-1}\\
&[h,x,y,b,c,[a,y]][h,x,y,c,[a,y],b][h,x,b,c,y,[a,y]][h,y,b,c,x,[a,y]]\\
&[h,x,[a,y],b,c,y][h,y,[a,y],b,c,x][h,x,c,y,[a,y],b][h,y,c,x,[a,y],b]\\
&[h,x,[a,y],b,y,c]^{-1}[h,y,[a,y],b,x,c]^{-1}[h,c,[a,y],y,x,b]^{-1}[h,b,c,[a,y],y,x]^{-1}z^{-12}.
\end{aligned}
$$

In terms of (A.1) this gives

$$
\begin{aligned}
&[h,b,c,x,x,y,[a,y]][h,c,x,x,y,[a,y],b][h,[a,y],b,c,x,x,y][h,[a,y],b,x,x,y,c]^{-1}\\
&[h,x,x,y,b,c,[a,y]][h,x,x,y,c,[a,y],b][h,x,x,b,c,y,[a,y]][h,y,b,c,x,x,[a,y]]\\
&[h,x,x,[a,y],b,c,y][h,y,[a,y],b,c,x,x][h,x,x,c,y,[a,y],b][h,y,c,x,x,[a,y],b]\\
&[h,x,x,[a,y],b,y,c]^{-1}[h,y,[a,y],b,x,x,c]^{-1}[h,c,[a,y],y,x,x,b]^{-1}\\
&[h,b,c,[a,y],y,x,x]^{-1}z^{-24} = z^4z^4z^4z^0zz^0z^{-1}zz^{-1}z^4z^{-6}z^{-5}z^5z^6z^4z^4z^{-24} = 1.
\end{aligned}
$$

Instead commuting by x gives

$$
\begin{aligned}
&[h,b,c,x,y,[a,y],x][h,c,x,y,[a,y],b,x][h,[a,y],b,c,x,y,x][h,[a,y],b,x,y,c,x]^{-1}\\
&[h,x,y,b,c,[a,y],x][h,x,y,c,[a,y],b,x][h,x,b,c,y,[a,y],x][h,y,b,c,x,[a,y],x]\\
&[h,x,[a,y],b,c,y,x][h,y,[a,y],b,c,x,x][h,x,c,y,[a,y],b,x][h,y,c,x,[a,y],b,x]\\
&[h,x,[a,y],b,y,c,x]^{-1}[h,y,[a,y],b,x,c,x]^{-1}[h,c,[a,y],y,x,b,x]^{-1}\\
&[h,b,c,[a,y],y,x,x]^{-1} = z^{-3}z^2z^{-3}z^{-1}zz^0z^{-1}z^{-2}z^{-1}z^4z^{-6}zz^5z^{-2}z^2z^4 = 1.
\end{aligned}
$$

Now consider $t = x[x,y]$. Then

$$\begin{aligned}
S =&[h,b,c,x,[x,y],[a,y]][h,c,x,[x,y],[a,y],b][h,[a,y],b,c,x,[x,y]]\\
&[h,[a,y],b,x,[x,y],c]^{-1}[h,x,[x,y],b,c,[a,y]][h,x,[x,y],c,[a,y],b]\\
&[h,x,b,c,[x,y],[a,y]][h,[x,y],b,c,x,[a,y]][h,x,[a,y],b,c,[x,y]]\\
&[h,[x,y],[a,y],b,c,x][h,x,c,[x,y],[a,y],b][h,[x,y],c,x,[a,y],b]\\
&[h,x,[a,y],b,[x,y],c]^{-1}[h,[x,y],[a,y],b,x,c]^{-1}[h,c,[a,y],[x,y],x,b]^{-1}\\
&[h,b,c,[a,y],[x,y],x]^{-1} = z^7z^7z^7z^0z^{-2}z^0z^{-5}z^{-5}z^{-2}z^7z^{-1}z^{-4}z^4zz^{-7}z^{-7} = 1.
\end{aligned}$$

Next suppose that $u = [a,x,y]$. Then, for $t = x$,

$$\begin{aligned}
S =&[h,b,c,x,[a,x,y]][h,c,x,[a,x,y],b][h,[a,x,y],b,c,x][h,[a,x,y],b,x,c]^{-1}\\
&[h,x,b,c,[a,x,y]][h,x,c,[a,x,y],b][h,c,[a,x,y],x,b][h,b,c,[a,x,y],x].
\end{aligned}$$

Using Section A.3 this gives, in terms of (A.1),

$$\begin{aligned}
&([h,y,x,a,y,b,c,x]^3[h,y,a,x,y,b,c,x]^{-3}[h,x,c,b,y,x,a,y]^{-3}[h,x,c,b,y,a,x,y]^3)^3\\
=&(z^{-6}z^0z^0z^6)^3 = 1.
\end{aligned}$$

Instead commuting by y gives

$$\begin{aligned}
&[h,b,c,x,[a,x,y],y][h,c,x,[a,x,y],b,y][h,[a,x,y],b,c,x,y][h,[a,x,y],b,x,c,y]^{-1}\\
&[h,x,b,c,[a,x,y],y][h,x,c,[a,x,y],b,y][h,c,[a,x,y],x,b,y][h,b,c,[a,x,y],x,y]\\
=&z^{-3}z^4z^3z^{-5}z^3z^{-6}z^4z^0 = 1.
\end{aligned}$$

Now consider $t = xy$. Here

$$\begin{aligned}
S =&[h,b,c,x,y,[a,x,y]][h,c,x,y,[a,x,y],b][h,[a,x,y],b,c,x,y][h,[a,x,y],b,x,y,c]^{-1}\\
&[h,x,y,b,c,[a,x,y]][h,x,y,c,[a,x,y],b][h,x,b,c,y,[a,x,y]][h,y,b,c,x,[a,x,y]]\\
&[h,x,[a,x,y],b,c,y][h,y,[a,x,y],b,c,x][h,x,c,y,[a,x,y],b][h,y,c,x,[a,x,y],b]\\
&[h,x,[a,x,y],b,y,c]^{-1}[h,y,[a,x,y],b,x,c]^{-1}[h,c,[a,x,y],y,x,b]^{-1}\\
&[h,b,c,[a,x,y],y,x]^{-1} = z^3z^3z^3z^6z^0z^{-6}z^{-3}z^{-3}z^0z^3z^5z^4z^{-4}z^{-5}z^{-3}z^{-3} = 1.
\end{aligned}$$

Now suppose that $u = [a, y, x]$. Then, for $t = x$,

$$\begin{aligned} S =&[h, b, c, x, [a, y, x]][h, c, x, [a, y, x], b][h, [a, y, x], b, c, x][h, [a, y, x], b, x, c]^{-1}\\ &[h, x, b, c, [a, y, x]][h, x, c, [a, y, x], b][h, c, [a, y, x], x, b][h, b, c, [a, y, x], x]. \end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$\begin{aligned} &([h, x, y, a, y, b, c, x]^3[h, y, a, y, x, b, c, x]^{-3}[h, x, c, b, x, y, a, y]^{-3}[h, x, c, b, y, a, y, x]^3)^3\\ =&(z^3z^6z^{-6}z^{-3})^3 = 1. \end{aligned}$$

Instead commuting by y gives

$$\begin{aligned} &[h, b, c, x, [a, y, x], y][h, c, x, [a, y, x], b, y][h, [a, y, x], b, c, x, y][h, [a, y, x], b, x, c, y]^{-1}\\ &[h, x, b, c, [a, y, x], y][h, x, c, [a, y, x], b, y][h, c, [a, y, x], x, b, y][h, b, c, [a, y, x], x, y]\\ =&z^{-3}z^5z^0z^{-4}z^0z^{-6}z^5z^3 = 1. \end{aligned}$$

Now consider $t = xy$. Here

$$\begin{aligned} S =&[h, b, c, x, y, [a, y, x]][h, c, x, y, [a, y, x], b][h, [a, y, x], b, c, x, y][h, [a, y, x], b, x, y, c]^{-1}\\ &[h, x, y, b, c, [a, y, x]][h, x, y, c, [a, y, x], b][h, x, b, c, y, [a, y, x]][h, y, b, c, x, [a, y, x]]\\ &[h, x, [a, y, x], b, c, y][h, y, [a, y, x], b, c, x][h, x, c, y, [a, y, x], b][h, y, c, x, [a, y, x], b]\\ &[h, x, [a, y, x], b, y, c]^{-1}[h, y, [a, y, x], b, x, c]^{-1}[h, c, [a, y, x], y, x, b]^{-1}\\ &[h, b, c, [a, y, x], y, x]^{-1} = z^0z^0z^0z^6z^3z^{-6}z^{-3}z^{-3}z^3z^0z^4z^5z^{-5}z^{-4}z^0z^0 = 1. \end{aligned}$$

Now suppose that $u = a$, $v = [b, y]$ and $w = [b, y]$. For $t = x$,

$$\begin{aligned} S =&[h, [b, y], [c, y], x, a][h, [c, y], x, a, [b, y]][h, a, [b, y], [c, y], x][h, a, [b, y], x, [c, y]]^{-1}\\ &[h, x, [b, y], [c, y], a][h, x, [c, y], a, [b, y]][h, [c, y], a, x, [b, y]][h, [b, y], [c, y], a, x]\\ &[h, x, [b, y], [c, y], x, a][h, x, a, [b, y], [c, y], x][h, x, [c, y], x, a, [b, y]]\\ &[h, x, a, [b, y], x, [c, y]]^{-1}[h, [c, y], a, x, x, [b, y]]^{-1}[h, [b, y], [c, y], a, x, x]^{-1}. \end{aligned}$$

In terms of (A.1) this gives, using (A.2),

$$\begin{aligned} &[h, a, [b, y], [c, y], x, x]^3[h, x, x, [c, y], [b, y], a]^{-3}[h, x, [b, y], [c, y], x, a]^2\\ &[h, x, a, [b, y], [c, y], x]^2[h, x, [c, y], x, a, [b, y]]^2[h, x, a, [b, y], x, [c, y]]^{-2}\\ &[h, [c, y], a, x, x, [b, y]]^{-2}[h, [b, y], [c, y], a, x, x]^{-2} = z^{-18}z^{-18}z^{24}z^{-12}z^4z^{-4}z^{12}z^{12} = 1. \end{aligned}$$

Instead commuting by x gives

$$\begin{aligned}
&[h,[b,y],[c,y],x,a,x][h,[c,y],x,a,[b,y],x][h,a,[b,y],[c,y],x,x][h,a,[b,y],x,[c,y],x]^{-1}\\
&[h,x,[b,y],[c,y],a,x][h,x,[c,y],a,[b,y],x][h,[c,y],a,x,[b,y],x][h,[b,y],[c,y],a,x,x]\\
=&z^{12}z^{2}z^{-6}z^{-2}z^{-6}z^{4}z^{2}z^{-6}=1.
\end{aligned}$$

Finally we consider the case u, v and w are all of different forms. First consider $u=a$, $v=[b,x]$ and $w=[c,y]$. When $t=x$,

$$\begin{aligned}
S=&[h,[b,x],[c,y],x,a][h,[c,y],x,a,[b,x]][h,a,[b,x],[c,y],x][h,a,[c,y],x,[b,x]]\\
&[h,x,[c,y],[b,x],a]^{-1}[h,x,[b,x],a,[c,y]]^{-1}[h,[b,x],a,x,[c,y]]^{-1}\\
&[h,[c,y],[b,x],a,x]^{-1}.
\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h,a,[b,x],y,c,y,x][h,x,y,c,y,[b,x],a]^{-1})^{-9}=(z^{-5}z^{5})^{-9}=1.$$

Instead commuting by y gives

$$\begin{aligned}
&[h,[b,x],[c,y],x,a,y][h,[c,y],x,a,[b,x],y][h,a,[b,x],[c,y],x,y][h,a,[c,y],x,[b,x],y]\\
&[h,x,[c,y],[b,x],a,y]^{-1}[h,x,[b,x],a,[c,y],y]^{-1}[h,[b,x],a,x,[c,y],y]^{-1}\\
&[h,[c,y],[b,x],a,x,y]^{-1}=zz^{-10}zz^{10}zz^{2}z^{-10}z^{5}=1.
\end{aligned}$$

Now suppose that $t=xy$. Then, by (A.2),

$$\begin{aligned}
S=&[h,a,[b,x],[c,y],x,y]^{3}[h,x,y,[c,y],[b,x],a]^{-3}[h,x,[b,x],[c,y],y,a]\\
&[h,y,[b,x],[c,y],x,a][h,x,a,[b,x],[c,y],y][h,y,a,[b,x],[c,y],x][h,x,[c,y],y,a,[b,x]]\\
&[h,y,[c,y],x,a,[b,x]][h,x,a,[c,y],y,[b,x]][h,y,a,[c,y],x,[b,x]][h,[b,x],a,x,y,[c,y]]\\
&[h,[b,x],a,y,x,[c,y]][h,[c,y],[b,x],a,x,y][h,[c,y],[b,x],a,y,x]\\
=&z^{3}z^{15}z^{-13}zz^{5}zz^{8}z^{-10}z^{-8}z^{10}z^{-5}z^{-1}z^{-5}z^{-1}=1.
\end{aligned}$$

Next consider $u=a$, $v=[b,y]$ and $w=[c,x]$. When $t=x$,

$$\begin{aligned}
S=&[h,[b,y],[c,x],x,a][h,[c,x],x,a,[b,y]][h,a,[b,y],[c,x],x][h,a,[c,x],x,[b,y]]\\
&[h,x,[c,x],[b,y],a]^{-1}[h,x,[b,y],a,[c,x]]^{-1}[h,[b,y],a,x,[c,x]]^{-1}\\
&[h,[c,x],[b,y],a,x]^{-1}.
\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h, a, y, b, y, [c, x], x][h, x, [c, x], y, b, y, a]^{-1})^{-9} = (z^{-7}z^{7})^{-9} = 1.$$

Instead commuting by y gives

$$\begin{aligned} S =&[h, [b, y], [c, x], x, a, y][h, [c, x], x, a, [b, y], y][h, a, [b, y], [c, x], x, y][h, a, [c, x], x, [b, y], y]\\ &[h, x, [c, x], [b, y], a, y]^{-1}[h, x, [b, y], a, [c, x], y]^{-1}[h, [b, y], a, x, [c, x], y]^{-1}\\ &[h, [c, x], [b, y], a, x, y]^{-1} = z^{-13}z^{8}z^{5}z^{-8}z^{5}z^{-6}z^{8}z = 1. \end{aligned}$$

Now suppose that $t = xy$. Note that the weight six part of R is the same as in the case $v = [b, x]$ and $w = [c, y]$, by swapping x and y. Thus we need only check that (A.2) is the same. Indeed

$$[h, a, [b, y], [c, x], x, y]^{3}[h, x, y, [c, x], [b, y], a]^{-3} = z^{15}z^{3} = z^{18}$$

is the same and so this finishes this case.

Next consider the case $u = [a, x]$, $v = b$ and $w = [c, y]$. When $t = x$,

$$\begin{aligned} S =&[h, b, [c, y], x, [a, x]][h, [c, y], x, [a, x], b][h, [a, x], b, [c, y], x][h, [a, x], [c, y], x, b]\\ &[h, x, [c, y], b, [a, x]]^{-1}[h, x, b, [a, x], [c, y]]^{-1}[h, b, [a, x], x, [c, y]]^{-1}\\ &[h, [c, y], b, [a, x], x]^{-1}. \end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h, [a, x], b, y, c, y, x][h, x, y, c, y, b, [a, x]]^{-1})^{-9} = (z^{2}z^{-2})^{-9} = 1.$$

Instead commuting by y gives

$$\begin{aligned} &[h, b, [c, y], x, [a, x], y][h, [c, y], x, [a, x], b, y][h, [a, x], b, [c, y], x, y][h, [a, x], [c, y], x, b, y]\\ &[h, x, [c, y], b, [a, x], y]^{-1}[h, x, b, [a, x], [c, y], y]^{-1}[h, b, [a, x], x, [c, y], y]^{-1}\\ &[h, [c, y], b, [a, x], x, y]^{-1} = z^{-10}z^{8}z^{-6}z^{-1}z^{-6}z^{5}z^{8}z^{2} = 1. \end{aligned}$$

Now suppose that $t = xy$. Then, by (A.2),

$$\begin{aligned}S =&[h,[a,x],b,[c,y],x,y]^3[h,x,y,[c,y],b,[a,x]]^{-3}[h,x,b,[c,y],y,[a,x]]\\&[h,y,b,[c,y],x,[a,x]][h,x,[a,x],b,[c,y],y][h,y,[a,x],b,[c,y],x][h,x,[c,y],y,[a,x],b]\\&[h,y,[c,y],x,[a,x],b][h,x,[a,x],[c,y],y,b][h,y,[a,x],[c,y],x,b][h,b,[a,x],x,y,[c,y]]\\&[h,b,[a,x],y,x,[c,y]][h,[c,y],b,[a,x],x,y][h,[c,y],b,[a,x],y,x]\\=&z^{-18}z^6z^8z^{-10}z^2z^{-6}z^{-10}z^8z^{13}z^{-1}z^{-2}z^6z^{-2}z^6 = 1.\end{aligned}$$

Next consider $u = [a, y]$, $v = b$ and $w = [c, x]$. When $t = x$,

$$\begin{aligned}S =&[h,b,[c,x],x,[a,y]][h,[c,x],x,[a,y],b][h,[a,y],b,[c,x],x][h,[a,y],[c,x],x,b]\\&[h,x,[c,x],b,[a,y]]^{-1}[h,x,b,[a,y],[c,x]]^{-1}[h,b,[a,y],x,[c,x]]^{-1}\\&[h,[c,x],b,[a,y],x]^{-1}.\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h,y,a,y,b,[c,x],x][h,x,[c,x],b,y,a,y]^{-1})^{-9} = (z^{-2}z^2)^{-9} = 1.$$

Instead commuting by y gives

$$\begin{aligned}S =&[h,b,[c,x],x,[a,y],y][h,[c,x],x,[a,y],b,y][h,[a,y],b,[c,x],x,y][h,[a,y],[c,x],x,b,y]\\&[h,x,[c,x],b,[a,y],y]^{-1}[h,x,b,[a,y],[c,x],y]^{-1}[h,b,[a,y],x,[c,x],y]^{-1}\\&[h,[c,x],b,[a,y],x,y]^{-1} = z^8z^{-10}z^2z^{13}z^2zz^{-10}z^{-6} = 1.\end{aligned}$$

Now suppose that $t = xy$. Note that the weight six part of R is the same as in the case $u = [a, x]$ and $w = [c, y]$, by swapping x and y. Thus we need only check that (A.2) is the same. Indeed

$$[h,[a,y],b,[c,x],x,y]^3[h,x,y,[c,x],b,[a,y]]^{-3} = z^6z^{-18} = z^{-12}$$

is the same and so this finishes this case.

Next consider the case $u = [a, x]$, $v = [b, y]$ and $w = c$. When $t = x$,

$$\begin{aligned}S =&[h,[b,y],c,x,[a,x]][h,c,x,[a,x],[b,y]][h,[a,x],[b,y],c,x][h,[a,x],c,x,[b,y]]\\&[h,x,c,[b,y],[a,x]]^{-1}[h,x,[b,y],[a,x],c]^{-1}[h,[b,y],[a,x],x,c]^{-1}\\&[h,c,[b,y],[a,x],x]^{-1}.\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h,[a,x],y,b,y,c,x][h,x,c,y,b,y,[a,x]]^{-1})^{-9} = (z^7z^{-7})^{-9} = 1.$$

Instead commuting by y gives

$$\begin{aligned}&[h,[b,y],c,x,[a,x],y][h,c,x,[a,x],[b,y],y][h,[a,x],[b,y],c,x,y][h,[a,x],c,x,[b,y],y]\\&[h,x,c,[b,y],[a,x],y]^{-1}[h,x,[b,y],[a,x],c,y]^{-1}[h,[b,y],[a,x],x,c,y]^{-1}\\&[h,c,[b,y],[a,x],x,y]^{-1} = z^8z^{-13}zz^{10}zzz^{-13}z^5 = 1.\end{aligned}$$

Now suppose that $t = xy$. Then, by (A.2),

$$\begin{aligned}S =&[h,[a,x],[b,y],c,x,y]^3[h,x,y,c,[b,y],[a,x]]^{-3}[h,x,[b,y],c,y,[a,x]]\\&[h,y,[b,y],c,x,[a,x]][h,x,[a,x],[b,y],c,y][h,y,[a,x],[b,y],c,x][h,x,c,y,[a,x],[b,y]]\\&[h,y,c,x,[a,x],[b,y]][h,x,[a,x],c,y,[b,y]][h,y,[a,x],c,x,[b,y]][h,[b,y],[a,x],x,y,c]\\&[h,[b,y],[a,x],y,x,c][h,c,[b,y],[a,x],x,y][h,c,[b,y],[a,x],y,x]\\=&z^3z^{15}z^{-10}z^8z^5zzz^{-13}z^{-8}z^{10}z^{-5}z^{-1}z^{-5}z^{-1} = 1.\end{aligned}$$

Finally consider $u = [a,y]$, $v = [b,x]$ and $w = c$. When $t = x$,

$$\begin{aligned}S =&[h,[b,x],c,x,[a,y]][h,c,x,[a,y],[b,x]][h,a,[b,x],c,x][h,[a,y],c,x,[b,x]]\\&[h,x,c,[b,x],[a,y]]^{-1}[h,x,[b,x],[a,y],c]^{-1}[h,[b,x],[a,y],x,c]^{-1}\\&[h,c,[b,x],[a,y],x]^{-1}.\end{aligned}$$

Using Section A.3 and (A.2) this gives, in terms of (A.1),

$$([h,y,a,y,[b,x],c,x][h,x,c,[b,x],y,a,y]^{-1})^{-9} = (z^5z^{-5})^{-9} = 1.$$

Instead commuting by y gives

$$\begin{aligned}S =&[h,[b,x],c,x,[a,y],y][h,c,x,[a,y],[b,x],y][h,[a,y],[b,x],c,x,y][h,[a,y],c,x,[b,x],y]\\&[h,x,c,[b,x],[a,y],y]^{-1}[h,x,[b,x],[a,y],c,y]^{-1}[h,[b,x],[a,y],x,c,y]^{-1}\\&[h,c,[b,x],[a,y],x,y]^{-1} = z^{-10}zz^5z^{-8}z^5z^5zz = 1.\end{aligned}$$

Now suppose that $t = xy$. Note that the weight six part of R is the same as in the case $u = [a,x]$ and $v = [b,y]$, by swapping x and y. Thus we need only check that (A.2) is the same. Indeed

$$[h,[a,y],[b,x],c,x,y]^3[h,x,y,c,[b,x],[a,y]]^{-3}=z^{15}z^3=z^{18}$$

is the same and so this finishes this case and the proof of (v).

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